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ANALYTICAL STUDY OF CERTAIN ASPECTS OF UNDERWATER SHOCKWAVE PROPAGATION : Sloping Bottom, Icecap, Sedimentary Bottom

David Feit and William Thompson, Jr.

October 1966

Final Report U-246-174
Prepared for Office of Naval Research
Field Projects Branch, Earth Sciences Division
Contract Nonr-3939(00), NR 321-002/8-27-64

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Abstract

Various problems concerning the effects of the boundaries of the ocean on the propagation of pressure waves in the ocean are considered. The propagation of a transient pressure wave in a wedge shaped region of fluid is treated. This is the model chosen to describe the situation in which an underwater explosion takes place in a coastal ocean region which is characterized by a strongly sloping bottom. In an attempt to study the effects of the polar ice cap on the propagation of a pressure wave, the reflection of a plane wave onto a rough boundary separating a fluid half space and a thick fluid layer of differing sound speed and density is considered. These results are currently being used to construct the response to a transient pressure pulse and to generate numerical results for conditions representative of underwater explosions. The final section presents numerical values of the reflection coefficient as a function of grazing angle for the case of a plane wave incident on a porous elastic bottom. The analytical expressions used were derived in an earlier report of this series.

Table of Contents

List of Figures

I. INTRODUCTION

The underwater detonation of an explosive charge, whether it be nuclear or conventional, produces a region of high pressure. This pressure is transmitted to the surrounding water and propagates as a shockwave in all directions. It is well known that the geometric and mechanical properties of the boundary media as well as the propagating media itself can strongly modify the pressure pulse that is sensed at a given location. The study of such effects is necessary for an understanding of underwater explosion effects and explosive echo-ranging.

The actual situation is of course quite complex. In the immediate vicinity of the explosion non-linear effects are important due to the extremely high pressures generated. The pressure pulse propagates through water which is inhomogeneous, the sound speed and density varying with depth. The boundary media can themselves be quite complicated in their mechanical behavior and furthermore the interface between the water and the boundary is often highly irregular. In order to make any mathematical analysis tractable many simplifying assumptions must be made. For the purposes of this report we restrict ourselves to analyzing the pressure field in terms of linear acoustic theory, i.e., at distances large enough from the source of the explosion so that the pressure amplitude has decreased sufficiently to permit the use of the linearized equations of motion. The fluid through which the pressure pulse propagates is assumed homogeneous.

An earlierreport[1] concerned itself mainly with the effects of the mechanical properties of the boundary mdeia, treating the boundary media as a liquid saturated porous elastic solid (mathematical model of a consolidated sedimentary bottom). In this study the interface between the bottom media and the water was taken as a smooth plane parallel to the free surface of the ocean.

The present report, for the most part, will emphasize the geometric effects of the boundaries rather than the mechanical behavior of the boundary media. Sect. II treats the case of a pressure pulse propagating in a coastal region characterized by a strongly sloping bottom. Sect. III concerns itself with the reflection of plane pressure waves from a rough boundary while Sec. IV treats the reflection of waves from sinusoidal boundaries between various types of media. These results are currently being used to find the response to a transient pressure wave and to generate numerical results for conditions representative of underwater explosions. Finally, Sec. V. presents numerical results for the reflection of a

plane wave incident from a liquid onto a liquid saturated porous elastic solid. (The theory of this was presented by Eichler and Rattayya.[1])

II. PRESSURE PULSE PROPAGATION FROM AN UNDERWATER EXPLOSION IN THE VICINITY OF A SLOPING BOTTOM

In recent years interest has developed in the propagation of pressure waves from underwater explosions taking place in coastal ocean regions. These areas are characterized by a strongly sloping bottom consisting of volcanic or coral rock. The angle that the bottom makes with the horizontal varies from about 10° to as much as 40°.[2] The situation that these regions present is quite different from that of either the continental shelves or the deep ocean basins, where the water depth can be considered to be fairly constant.

In order to study the problem of underwater explosions in the vicinity of strongly sloping bottoms, the situation is idealized by considering a transient point or line source situated in a wedge shaped region occupied by a slightly compressible, non-viscous liquid of density ρ, and constant sound speed c. Cylindrical coordinates (r,θ,z) are introduced such that the z-axis coincides with the edge of the wedge and the boundary walls of the wedge are given by $\theta=0$, and $\theta=\psi$. The boundary at $\theta=0$ is taken to represent the ocean surface, while the boundary at $\theta=\psi$ will correspond to the ocean bottom. The point source is located by the cylindrical coordinates $(r_0,\theta_0,0)$ as shown in Fig. 1. In contrast to the problem of a point source located in an ocean of constant depth, which benefits from cylindricalsymmetry, the problem of a point source in a wedge shaped region is truly a three-dimensional problem.

A rigorous mathematical model for the above situation should assume that the bottom boundary $\theta=\psi$ is in contact with an elastic solid or another liquid of different density and sound speed, but this problem has been shown to be mathematically intractable. Kearsley's thesis gives a detailed description of the difficulties involved.[3] Therefore as a first approximation to the problem the ocean bottom will be assumed to be infinitely rigid, i.e., the normal derivative of the pressure vanishes at $\theta=\psi$, and the ocean surface will be taken as a free surface, i.e., the pressure is zero at the surface $\theta=0$. The method used will very closely follow that of Oberhettinger.[4] This reference also contains an excellent bibliography of earlier work on the problem of wave diffraction by wedges. The present work goes beyond Oberhettinger, in that it solves the problem with one boundary rigid and one boundary free, whereas Oberhettinger considers the case where both boundaries are either free or rigid.

A. Line Source Excitation

First consider the case of a time dependent line source parallel to the z-direction situated on the line $S(r_o,\theta_o)$ within the wedge. The pressure field within the wedge $0 < \theta < \psi$ must then satisfy the wave equation

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \frac{-\delta(r-r_o)\delta(\theta-\theta_o)f(t)}{r} \tag{II.1}$$

where $f(t)$ represents the time history of the source and $\delta(x)$ is a Dirac delta function, together with the boundary conditions

$$\left.\begin{aligned} p(r,\theta) &= 0 \\ \frac{\partial p}{\partial \theta}(r,\psi) &= 0 \end{aligned}\right. \tag{II.2}$$

Taking a Laplace transform in time of both sides of Eq. II.1 results in

$$\frac{\partial^2 \bar{p}}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \bar{p}}{\partial \theta^2} - \frac{s^2}{c^2}\bar{p} = -\frac{\delta(r-r_o)\delta(\theta-\theta_o)F(s)}{r} \tag{II.3}$$

where

$$\left.\begin{aligned} \bar{p}(r,\theta) &= \int_0^\infty p(r,\theta,t)^{-st}dt \\ p(r,\theta,t) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \bar{p}(r,\theta)e^{st}\,ds \end{aligned}\right\} \tag{II.4}$$

and

$$F(s) = \int_0^\infty f(t)e^{st}\,dt \tag{II.5}$$

is the Laplace transform of the time dependent source strength. In order to solve Eq. II.3 together with the transformed boundary conditions corresponding to Eq. II.2, consider first the following problem; a transient source is located at $r=r_o,\theta=\theta_o$ in a wedge shaped region $0 < \theta < \alpha$, the boundaries of which are pressure-release surfaces, i.e., the pressure vanishes at $\theta=0$ and at $\theta=\alpha$. The solution to this problem can be easily derived (see Appendix A) and is given by

$$\bar{p}(r,\theta) = \frac{F(s)}{2\alpha}\sum_{n=0}^{\infty} \epsilon_n[\cos\nu_n(\theta-\theta_o)-\cos\nu_n(\theta+\theta_o)]\,I_{\nu_n}(\tfrac{s}{c}\,r<)\,K_{\nu_n}(\tfrac{s}{c}r>) \tag{II.6}$$

where $r<$ and $r>$ are, respectively, the smaller or larger of the quantities r and r_o, $\nu_n = \frac{n\pi}{\alpha}$ and $\epsilon_o = 1$, $\epsilon_n = 2$ for $n \geq 1$. The solution for an image source located at $r=r_o$, $\theta=\alpha-\theta_o$ is

$$\bar{p}(r,\theta) = \frac{F(s)}{2\alpha} \sum_{n=0}^{\infty} \epsilon_n[\cos\nu_n(\theta-\alpha+\theta_o)-\cos\nu_n(\theta-\theta_o+\alpha)]I_{\nu_n}(\frac{s}{c}\, r<)K_{\nu_n}(\frac{s}{c}\, r>) \tag{II.7}$$

Combining the two solutions and letting $\alpha = 2\psi$ one arrives at the Laplace transform of the solution to the problem posed by Eqs. II.1 and II.2, and this can be written as

$$\begin{aligned} \bar{p}(r,\theta) = \frac{F(s)}{4\psi} \sum_{n=0}^{\infty} \epsilon_n[&\cos\nu_n(\theta-\theta_o) - \cos\nu_n(\theta+\theta_o) \\ &+ \cos\nu_n(\theta+\theta_o-2\psi)-\cos\nu_n(\theta-\theta_o+2\psi)]I_{\nu_n}(\frac{s}{c}\, r<)K_{\nu_n}(\frac{s}{c}\, r>) \end{aligned} \tag{II.8}$$

For convenience in further manipulation define the function $\bar{G}$ by

$$\bar{G}(r,\theta;r_o,\theta_o) = \frac{1}{4\psi} \sum_{n=0}^{\infty} \epsilon_n[\cos\nu_n(\theta-\theta_o) - \cos\nu_n(\theta+\theta_o)]I_{\nu_n}(\frac{s}{c}\, r<)K_{\nu_n}(\frac{s}{c}\, r>) \tag{II.9}$$

so that

$$\bar{p}(r,\theta) = \{\bar{G}(r,\theta;r_o,\theta_o) + \bar{G}(r,\theta;r_o,2\psi-\theta_o)\}\, F(s) \tag{II.10}$$

Following Oberhettinger[4] it can be shown that $\bar{G}$ can be put in the form

$$\bar{G}(r,\theta;r_o,\theta_o) = \bar{L}(r,r_o,\theta-\theta_o) - \bar{L}(r,r_o,\theta+\theta_o) \tag{II.11}$$

where

$$L(r,r_o\varphi) = \frac{1}{2\pi}\sum_{n=n_1}^{n_2} K_o\{\frac{s}{c}[r^2+r_o^2 - 2\,rr_o\cos(4n\psi+\varphi)]^{\frac{1}{2}}\}$$

$$\frac{-1}{8\psi\pi}\int_0^\infty K_o[\frac{s}{c}(r^2+r_o^2 + 2\,rr_o\cosh x)^{\frac{1}{2}}]\times \tag{II.12}$$

$$\left\{\frac{\sin\frac{\pi}{2\psi}(\pi-\varphi)}{\cosh(\frac{\pi x}{2\psi})-\cos[\frac{\pi}{2\psi}(\pi-\varphi)]} + \frac{\sin[\frac{\pi}{2\psi}(\pi+\varphi)]}{\cosh(\frac{\pi x}{2\psi})-\cos[\frac{\pi}{2\psi}(\pi+\varphi)]}\right\}dx$$

where n_1 is the largest positive or negative number which is smaller than or equal to $-(\pi+\varphi/4\psi)$ and n_2 is the largest positive or negative number which is smaller than or equal to $(\pi-\varphi/4\psi)$. If n_1 is greater than n_2 the sum in Eq. II.12 is zero.

B. Point Source Excitation

The line source solution derived in the previous section can be used to derive the solution for a point source. Using cylindrical coordinates (r,θ,z) the point source is located at $(r_o,\theta_o,0)$. Eq. II.1 now becomes

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2} + \frac{\partial^2 p}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\delta(r-r_o)\delta(\theta-\theta_o)\delta(z)f(t)}{r} \tag{II.13}$$

Taking a Fourier Cosine transform in z and a Laplace transform in time gives

$$\frac{\partial^2\bar{\bar{p}}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\bar{p}}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\bar{\bar{p}}}{\partial\theta^2} - (\frac{s^2}{c^2}+\lambda^2)\bar{\bar{p}} = -\frac{\delta(r-r_o)\delta(\theta-\theta_o)F(s)}{2r} \tag{II.14}$$

where λ is the Fourier Cosine transform parameter. It is then obvious that one can derive the point source solution from the line source solution by replacing s/c by $(s^2/c^2+\lambda^2)^{\frac{1}{2}}$ in the line source solution, multiplying each term of the solution by $(1/\pi\cos\lambda z)$ and integrating with respect to λ from 0 to ∞. In performing the above operation one can use the following well-known relationship[3]

$$(r^2+z^2)^{-\frac{1}{2}}\exp[\frac{s}{c}(r^2+z^2)^{\frac{1}{2}}] = \frac{2}{\pi}\int_0^\infty K_o[r(\frac{s^2}{c^2}+\lambda^2)^{\frac{1}{2}}]\cos\lambda z\,d\lambda \tag{II.15}$$

Operating on Eq. II.12 in the above described manner we obtain

$$\bar{L}(\theta) = \frac{1}{4\pi} \sum_{n=n_1}^{n_2} \frac{\exp\{-\frac{s}{c}[r^2+r_o^2+z^2-2rr_o\cos(\theta+4n\psi)]^{\frac{1}{2}}\}}{[r^2+r_o^2+z^2-2rr_o\cos(\theta+4n\psi)]^{\frac{1}{2}}}$$

$$- \frac{1}{16\psi\pi} \int_0^\infty H(x) \left[\frac{\sin[\frac{\pi}{2\psi}(\pi-\theta)]}{\cosh(\frac{\pi x}{2\psi})-\cos[\frac{\pi}{2\psi}(\pi-\theta)]} + \frac{\sin[\frac{\pi}{2\psi}(\pi+\theta)]}{\cosh(\frac{\pi x}{2\psi})-\cos[\frac{\pi}{2\psi}(\pi+\theta)]} \right] dx \qquad \text{(II.16)}$$

where

$$H(x) = \frac{\exp[-\frac{s}{c}(r^2+r_o^2+z^2+2rr_o\cosh x)^{\frac{1}{2}}]}{(r^2+r_o^2+z^2+2rr_o\cosh x)^{\frac{1}{2}}} \qquad \text{(II.17)}$$

Finally using Eqs. II.10 and II.11 $\bar{p}(r,\theta)$ can be written in the form

$$\bar{p}(r,\theta) = [\bar{L}(r,r_o,\theta-\theta_o)-\bar{L}(r,r_o,\theta+\theta_o) + \bar{L}(r,r_o,\theta-2\psi+\theta_o)-\bar{L}(r,r_o,\theta+2\psi-\theta_o)]\, F(s) \qquad \text{(II.18)}$$

where $\bar{L}(r,r_o,\theta)$ is defined by Eq. II.16. Each $\bar{L}(r,r_o,\theta)$ consists of two kinds of terms, one a sum of terms each of which has the same functional dependence as the point source but located at image points of the source with respect to the boundaries of the wedge. The number of images depends on the numbers n_1 and n_2. The other term is given by an integral expression which vanishes whenever the angle of the wedge ψ is a submultiple of π, i.e., $\psi = \pi/m$, $m = 1,2....$ If this is the case the solution is then given by the effect of the original source and a finite number of image sources which are necessary to satisfy the specific boundary conditions. The integral expression has been called the edge diffracted wave by Biot and Tolstoy.[5]

C. Transient Solution

It has been assumed in writing Eqs. II.1 and II.13 that both the line and point source are time-dependent, and that their time-dependence is given by $f(t)$. In aoplying the Laplace transform to the function $f(t)$ it has been assumed that $f(t)$ vanishes for $t < 0$. What remains to be shown is how the solutions depend on $f(t)$.

In order to present results for the time-dependent source it is convenient to introduce the following notation

$$R_n(\theta) = \{r^2+r_o^2+z^2-2\,rr_o\cos(\theta+4n\psi)\}^{\frac{1}{2}} \qquad \text{(II.19)}$$

$$R_n(x) = \{r^2+r_o^2+z^2+2\ rr_o\ \cosh x\}^{\frac{1}{2}} \tag{II.20}$$

$$g(x,\theta) = \frac{\sin[\frac{\pi}{2\psi}(\pi-\theta)]}{\cosh(\frac{\pi x}{2\psi})-\cos[\frac{\pi}{2\psi}(\pi-\theta)]} + \frac{\sin[\frac{\pi}{2\psi}(\pi+\theta)]}{\cosh(\frac{\pi x}{2\psi})-\cos[\frac{\pi}{2\psi}(\pi+\theta)]} \tag{II.21}$$

Using the above notation $\bar{L}(r,r_o,\theta)$ can be written as

$$\bar{L}(r,r_o,\theta) = \frac{1}{4\pi}\sum_{n=n_1}^{n_2} \frac{e^{-s/c\ R_n(\theta)}}{R_n(\theta)} - \frac{1}{16\psi\pi}\int_0^\infty \frac{e^{-s/c\ R_n(x)}}{R_n(x)}\ g(x,\theta)\ dx \tag{II.22}$$

The solution corresponding to the case $F(s)=1$, where $F(s)$ as given by Eq. II.5 is the Laplace transform of the function $f(t)$, represents the response of the system to a Dirac delta excitation applied at the time $t=0$.

This response is given by

$$p_\delta(r,\theta,t) = \frac{1}{4\pi}\sum_{n=1}^{n_2}\Bigg\{\frac{\delta[t-\frac{R_n(\theta-\theta_o)}{c}]}{R_n(\theta-\theta_o)} - \frac{\delta[t-\frac{R_n(\theta+\theta_o)}{c}]}{R_n(\theta+\theta_o)}$$
$$+\frac{\delta[t-\frac{R_n(\theta-2\psi+\theta_o)}{c}]}{R_n(\theta-2\psi+\theta_o)} - \frac{\delta[t-\frac{R_n(\theta+2\psi-\theta_o)}{c}]}{R_n(\theta+2\psi-\theta_o)} \tag{II.23}$$

$$-\frac{1}{16\psi\pi}\int_0^{x_1}\frac{\delta[t-\frac{R_n(x)}{c}]}{R_n(x)}\{g(x,\theta-\theta_o)-g(x,\theta+\theta_o)+g(x,\theta-2\psi+\theta_o)-g(x,\theta+2\psi-\theta_o)\}\ dx \tag{II.24}$$

where

$$x_1 = \cosh^{-1}\left(\frac{c^2t^2-r^2-r_o^2-z^2}{2rr_o}\right)\ . \tag{II.25}$$

For any given $f(t)$ the first four summations can be obtained explicitly while the integral contribution can be evaluated numerically.

III. SCATTERING OF A PLANE WAVE BY A ROUGH NON-SINUSOIDAL BOUNDARY BETWEEN A SEMI-INFINITE FLUID MEDIUM AND A THICK FLUID LAYER

The problem considered in this section is that of a plane wave scattered by a rough non-sinusoidal boundary between a semi-infinite fluid and a thick fluid layer. This is the first step necessary for the consideration of the scattering of a transient pressure pulse by the Arctic icecap. These results are currently being used to construct the response to a transient pressure pulse and to generate results for conditions representative of underwater explosions.

The reflection of waves from a periodic boundary has been studied quite extensively and for a discussion of this problem see the following section. Reflection of waves from arbitrary non-sinusoidal boundaries has not been studied so extensively. Recently Abubakar[6] and Dunkin and Eringen[7] have studied the reflection of waves from an arbitrary non-sinusoidal surface by means of a perturbation method. It is essentially this method which is used in the present analysis. Consider the case of a plane wave, propagating through a fluid of density ρ_o and sound speed c_o, incident on a liquid layer of density ρ_1, sound speed c_1 and of mean height h. The interface between the fluid and the liquid layer is taken to be $z=\epsilon f(x)$ where ϵ is a small parameter and $f(x)$ is the arbitrary function of x. The fluid lies in the half space $z > \epsilon f(x)$, while the liquid layer, whose upper boundary ($z= -h$) is taken to be planar, then lies in $-h < z < \epsilon f(x)$ (Fig. 2), where z is measured positive into the fluid half space.

As before, a plane sound wave of unit amplitude is assumed to be obliquely incident upon the irregular boundary separating the two fluids. The velocity potential φ_i of this wave is given by

$$\varphi_i = e^{i(\sigma_o x-\mu_o z)} \tag{III.1}$$

where $\sigma_o=k_o\sin\theta_o$, $\mu_o=k_o\cos\theta_o$, $(k_o=\omega/c_o)$, the projections of the incident wave number on the x- and z-axis respectively, θ_o is the angle that the plane wave normal makes with the z-axis as shown in Fig. 2.

The reflected wave φ_r is assumed to be of the form

$$\varphi_r = R\, e^{i(\sigma_o x-\mu_o z)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\xi)\, e^{i(\gamma_o z+\xi x)}\, d\xi \tag{III.2}$$

where $\gamma_o = \sqrt{k_o^2-\xi^2}$. In the case of a sinusoidal boundary the incident wave excites a denumerably infinite set of plane reflected waves whereas in this case it is assumed to scatter a continuous spectrum of plane waves.

The transmitted wave φ_t is assumed to be of a similar form and is given by

$$\varphi_t = T[e^{i(\sigma_1 x-\mu_1 z)} + C\, e^{i(\sigma_1 x+\mu_1 z)}] + \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\xi)[e^{i(\xi x-\gamma_1 z)} + D\, e^{i(\xi x+\gamma_1 z)}]\, d\xi \tag{III.3}$$

where θ_1 is the angle of refraction of the transmitted wave, $\sigma_1=k_1 \sin\theta_1$, $\mu_1=k_1\cos\theta_1$ are the projections of the transmitted wave number on the x, and z axes respectively, and $\gamma_1=\sqrt{k_1^2-\xi^2}$.

We assume that the upper surface of the liquid layer z= -h is a pressure-release surface, i.e., the potential given by Eq. III.3 must vanish at z= -h. Using this condition we can solve for C and D in Eq. III.3 and we obtain

$$C = - e^{2i\mu_1 h}$$
$$D = - e^{2i\gamma_1 h} \tag{III.4}$$

Substituting these values back into (III.3) we obtain

$$\varphi_t = T\, e^{i(\sigma_1 x-\mu_1 z)}[1-e^{2i\mu_1(z+h)}] + \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\xi)\, e^{i(\xi x-\gamma_1 z)}[1-e^{2i\gamma_1(z+h)}]\, d\xi \tag{III.5}$$

The rough boundary separating the two fluids is given by

$$z = \epsilon f(x) \tag{III.6}$$

where ϵ is a small characteristic parameter (for example the ratio of the maximum amplitude of the boundary to the smaller of the two acoustic wavelengths $\lambda_o = \frac{2\pi c_o}{\omega}$ or $\lambda_1 = \frac{2\pi c_1}{\omega}$).

In order to determine the unknowns R,T,A(ξ), B(ξ) appearing in Eq. III.2 and Eq. III.5 the boundary conditions at the interface $z=\epsilon f(x)$ must be satisfied. The two boundary conditions are continuity of pressure and continuity of normal velocity. In terms of the potential functions the condition of continuity of pressure is

$$\rho_o(\varphi_i + \varphi_r) = \rho_1\varphi_t \quad ; \quad z = \epsilon f(x) \tag{III.7}$$

The components of the unit normal to the surface $z=\epsilon f(x)$ are $n_x = -\epsilon f'[1+(\epsilon f')^2]^{-\frac{1}{2}}$, $n_z=[1+(\epsilon f')^2]^{-\frac{1}{2}}$, where $f'=df/dx$, and so the normal velocity condition is written as

$$\frac{\partial(\varphi_i+\varphi_r)}{\partial z} - \epsilon f' \left(\frac{\partial\varphi_i}{\partial x} + \frac{\partial\varphi_r}{\partial x}\right) = \frac{\partial\varphi_t}{\partial z} - \epsilon f' \frac{\partial\varphi_t}{\partial x} \quad ; \quad z = \epsilon f(x) \tag{III.8}$$

As mentioned earlier ϵ is taken to be a small parameter and therefore we expand the functions $A(\xi)$ and $B(\xi)$ into power series in ϵ.

$$A(\xi) = \sum_{n=1}^{\infty} \epsilon^n A_n(\xi) = \epsilon A_1(\xi) + \epsilon^2 A_2(\xi) + \dots \tag{III.9}$$

$$B(\xi) = \sum_{n=1}^{\infty} \epsilon^n A_n(\xi) = \epsilon B_1(\xi) + \epsilon^2 B_2(\xi) + \dots \tag{III.10}$$

Substituting these expressions into Eq. III.2 and Eq. III.5 and then the resulting expressions into Eq. III.7 yields

$$\rho_o\{e^{i\sigma_o x}[e^{-i\mu_o\epsilon f}+Re^{i\mu_o\epsilon f}] + \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}\epsilon^n A_n e^{i(\gamma_o\epsilon f+\xi x)}\, d\xi\} =$$

$$\rho_1\{Te^{i(\sigma_1 x-\mu_1\epsilon f)}[1-e^{2i\mu_1(\epsilon f+h)}]+\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}\epsilon^n B_n e^{i(\sigma_1 x-\gamma_1\epsilon f)}[1-e^{2i\gamma_1(\epsilon f+h)}]\, d\xi\} \tag{III.11}$$

Now expanding the exponentials of the form $e^{\alpha\epsilon f}$ into a power series

$$e^{\alpha\epsilon f} = 1 + \frac{\alpha\epsilon f}{1!} + \frac{\alpha^2\epsilon^2 f^2}{2!} + \dots \tag{III.12}$$

and equating the coefficients of like powers of ϵ in Eq. III.11 we obtain the equations

$$\rho_o Re^{i\sigma_o x} - \rho_1 T(1-e^{2i\mu_1 h})\, e^{i\sigma_1 x} = -\rho_o e^{i\sigma_o x} \tag{III.13a}$$

$$\frac{\rho_o}{2\pi}\int_{-\infty}^{\infty} A_1(\xi)e^{i\xi x}\, d\xi - \frac{\rho_1}{2\pi}\int_{-\infty}^{\infty} B(\xi)\,(1-e^{2i\gamma_1 h})\, e^{i\xi x}\, d\xi$$

$$= i\, f[\rho_o\mu_o(1-R)\, e^{i\sigma_o x} - \rho_1\mu_1 T(1+e^{2i\mu_1 h})\, e^{i\sigma_1 x}], \text{ etc.} \tag{III.13b}$$

Eqs. III.13a and III.13b are determined from the zeroth and first order ϵ terms

in Eq. III.11, respectively. In order that Eq. III.13a be satisfied for all values of x it is necessary that $\sigma_o = \sigma_1$ which means that

$$\frac{\sin\theta_o}{c_o} = \frac{\sin\theta_1}{c_1} \tag{III.14}$$

which is the familiar expression of Snell's Law.

In the same manner one satisfies the normal velocity condition Eq. III.8 and obtains the following two equations corresponding to the zeroth and first order terms in ϵ.

$$\mu_o e^{i\sigma_o x} R + \mu_1(1 + e^{2i\mu_1 h})\, e^{i\sigma_1 x}\, T = \mu_o e^{i\sigma_o x} \tag{III.15a}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \gamma_o A_1 e^{i\xi x}\, d\xi + \frac{1}{2\pi}\int_{-\infty}^{\infty} \gamma_1(1+e^{2i\mu_1 h})\, B\, e^{i\xi x}\, d\xi$$
$$= (\sigma_o f' - i\mu_o^2 f)(1+R)\, e^{i\sigma_o x} - (\sigma_1 f' - i\mu_1^2 f)(1-e^{2i\mu_1 h})\, Te^{i\sigma_1 x} \tag{III.15b}$$

Making use of Eq. III.14, Eqs. III.13a and III.13b become

$$\rho_o R - \rho_1(1-e^{2i\mu_1 h})\, T = \rho_o \tag{III.16}$$

$$\mu_o R + \mu_1(1+e^{2i\mu_1 h})\, T = \mu_o \tag{III.17}$$

Solving for R and T we obtain

$$R = \frac{\frac{\rho_1}{\mu_1}(1-e^{2i\mu_1 h}) - \frac{\rho_o}{\mu_o}(1+e^{2i\mu_1 h})}{\frac{\rho_1}{\mu_1}(1-e^{2i\mu_1 h}) + \frac{\rho_o}{\mu_o}(1+e^{2i\mu_1 h})} \tag{III.18}$$

$$T = \frac{2\rho_o}{\mu_o}\, \frac{1}{\frac{\rho_1}{\mu_1}(1-e^{2i\mu_1 h}) + \frac{\rho_o}{\mu_o}(1+e^{2i\mu_1 h})} \tag{III.19}$$

In order to solve for the functions $A(\xi)$ and $B(\xi)$, multiply each of the Eqs. III.13b and III.15b by $e^{-i\xi' x}$ and then integrate over x from $-\infty$ to ∞, making use of the identity

$$\int_{-\infty}^{\infty} e^{i(\xi-\xi')x}\, dx = 2\pi\delta(\xi-\xi') \tag{III.20}$$

where $\delta(\xi)$ is the Dirac delta function. The transformed Eqs. III.13b and III.15b become, respectively

$$\rho_o A_1(\xi) - \rho_1(1-e^{2i\gamma_1 h})\, B_1(\xi) = \qquad \text{(III.21)}$$
$$i\,\bar{f}(\xi-\sigma_o)\{\rho_o\mu_o(1-R) - \rho_1\mu_1 T(1+e^{2i\mu_1 h})\}$$

$$\gamma_o A_1(\xi) + \gamma_1(1+e^{2i\gamma_1 h})\, B_1(\xi) = \qquad \text{(III.22)}$$
$$i\bar{f}(\xi-\sigma_o)\{[\sigma_o(\xi-\sigma_o)-\mu_o^2](1+R)-[\sigma_o(\xi-\sigma_o)-\mu_1^2](1-e^{2i\mu_1 h})\, T\}$$

where

$$\bar{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx \qquad \text{(III.23)}$$
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(\xi)\, e^{i\xi x}\, d\xi$$

Eqs. III.21 and III.22 can be solved for A_1 and B_1. The present analysis will only calculate the first order perturbation to the reflected wave and so we need only obtain A_1 which is given by

$$A_1(\xi) = 2i\,\bar{f}(\xi-\sigma_o)\{[1-\frac{i\rho_1\mu_o}{\rho_o\mu_1}\tan\mu_1 h][1-\frac{i\rho_o\gamma_o}{\rho_o\gamma_1}\tan\gamma_1 h]\}^{-1}\times \qquad \text{(III.24)}$$
$$\{\mu_o(1-\frac{\rho_1}{\rho_o}) + \frac{\rho_1}{\rho_o}\frac{\mu_o}{\mu_1}\frac{1}{\gamma_1}\tan\gamma_1 h\,\tan\mu_1 h[\sigma_o(\xi-\sigma_o)(1-\frac{\rho_1}{\rho_o})-\mu_1^2+\frac{\rho_1}{\rho_o}\mu_o^2]\}$$

Insertion of this value of A_1 into Eq. III.2 yields the first order perturbation to the reflected potential as

$$\varphi_r^{(1)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_1(\xi)\, e^{i(\gamma_o z+\xi x)}\, d\xi \qquad \text{(III.25)}$$

where $\gamma_o = \sqrt{k_o^2-\xi^2}$. This integral can be evaluated asymptotically, i.e., at distances from the origin that are large enough so that $k_o r \gg 1$, where $r=x^2+z^2$ and the result is (see Appendix A)

$$\varphi_r^{(1)} \sim \frac{k_o}{2\pi r}\, e^{i(k_o r-\frac{\pi}{4})}\, A_1(k_o\sin\theta)\cos\theta\ . \qquad \text{(III.26)}$$

From Eq. III.2 it is clear that the first order perturbation to the reflected potential is given by a cylindrically spreading wave.

IV. REFLECTION OF PLANE WAVES AT AN IRREGULAR BOUNDARY

The reflection of a plane sound wave from a non-planar interface between two media has been termed by Uretsky[8] a "marvelously complex problem." The problem was first attacked by Rayleigh[9] and has since apparently captured the fancy of many workers for there is a wealth of literature dealing with the problem. The bulk of this literature however is concerned with the simplest case which is that of a sinusoidal pressure-release boundary. This simplest case is trouble enough however, for its solution involves solving infinite sets of equations for which, of course, only approximate solutions have been obtained. There is much less literature dealing with the case of the non-perfectly reflecting boundary such as a sinusoidal boundary between two fluids of different characteristic impedance, no doubt because the problem is a degree more difficult than the perfectly reflecting boundary inasmuch as the amplitudes of the rays transmitted into the second fluid constitute a second infinite set of unknowns to be dealt with.

We will briefly mention the other analyses of reflection from a non-perfectly reflecting sinusoidal boundary. Miles[10] treats the case of oblique incidence upon a sinusoidal boundary between two semi-infinite fluid media in contact and both he and Grasyuk[11] also look at the case of oblique incidence upon the sinusoidal boundary between a semi-infinite fluid in contact with a semi-infinite elastic medium. In both these analyses, the amplitude of the undulations of the boundary is assumed to be small compared to a wavelength of sound while the slope of the boundary is considered to be non-negligible. However it has been pointed out by Lapin[12] that Grasyuk's work has errors. Rayleigh in his work took the opposite point of view, i.e., the slope of the boundary is assumed small while the amplitude of the undulations is not. Rayleigh's analysis is also available in Ref. 13. Rayleigh limited his attention to normal incidence.

In the analysis that follows, we extend Rayleigh's work to allow for oblique incidence. An approximate solution, entirely analogous to that of Rayleigh, is obtained under the conditions that the acoustic wavelength is much less than the wavelength of the undulations of the boundary and the slope of the boundary is negligible. We then treat in turn the problems of reflection from a sinusoidal boundary of a plane, obliquely incident wave between 1) two semi-infinite fluid media, 2) a semi-infinite fluid and a semi-infinite elastic medium, 3) a finite thickness fluid layer sandwiched between two different semi-infinite fluid

media, and 4) a finite thickness elastic layer sandwiched between two semi-infinite fluid media. For these four cases, an approximate solution is again obtained under the restriction just mentioned, and, for the last two cases, with the additional restriction that the thickness of the layer be much greater than the amplitude of the boundary undulations.

Before beginning the analysis, a word must be said about the controversy that exists regarding Rayleigh's assumption that the reflected and transmitted waves can be written as a superposition of plane waves, each propagating in a discrete direction. Although it is agreed that this is a valid representation for the scattered field at a large distance from the boundary, the controversy pertains to the question of whether this representation is valid near the boundary, that is, in the region between the extremes of the undulations. No rigorous proof that Rayleigh's assumption is incorrect has yet been published although Uretsky[14] presents a strong argument to that effect. Marsh,[15] on the other hand, has published an argument which he claims justifies the Rayleigh assumption. In the analysis that follows, we unashamedly and without providing any additional justification use the Rayleigh approach.

A. Scattering from A Sinusoidal Boundary between Two Semi-Infinite Fluid Media

The geometry of the problem is shown in Fig. 3. The notation used closely follows that of Rayleigh[9] or Grasyuk.[11] A plane sound wave of unit amplitude* (defined by the velocity potential φ_i) is assumed to be obliquely incident upon the sinusoidally rough boundary (it is assumed that there is no y-direction variation of the boundary undulations).

$$\varphi_i = \exp i(\sigma_o x - \mu_o z) \tag{IV.1}$$

where σ_o and μ_o are the projections of the incident wave number upon the x- and z-axes respectively, i.e., $\sigma_o = k_o \sin\theta_o$ and $\mu_o = k_o \cos\theta_o (k_o = \omega/c_o)$.

According to Rayleigh,[9] this incident wave excites a denumerably infinite set of plane reflected waves φ_r and of plane transmitted waves φ_1, which can be written

$$\varphi_r = \sum_{n=-\infty}^{\infty} A_n \exp\{i[(\sigma_o + gn)x + \mu_n z]\} \tag{IV.2}$$

$$\text{with } \mu_n^2 = k_o^2 - (\sigma_o + gn)^2 \tag{IV.3}$$

*The harmonic time factor $\exp(-i\omega t)$ will, as usual, be suppressed.

$$\varphi_1 = \sum_{n=-\infty}^{\infty} B_n \exp\{i[(\sigma_o + gn)x - \chi_n z]\} \tag{IV.4}$$

$$\text{with } \chi_n^2 = k_1^2 - (\sigma_o + gn)^2 \tag{IV.5}$$

where g is proportional to the reciprocal of the wavelength of the boundary undulations (see Fig. 3). Ref. 16 contains an excellent physical explanation for writing the reflected and transmitted waves in this fashion. The restriction Eq. IV.3 on the direction cosines of the reflected rays, μ_n, results from the fact that Eq. IV.2 must satisfy the wave equation in the incident fluid; the restriction IV.5 arises in an analogous fashion.

The ray defined by n=0 is the specularly reflected one, i.e., the one for which the angle of reflection equals the angle of incidence and the only one which would be present if the boundary were plane. The rays defined by $n > 0$ are sometimes called forward scattered, while those for $n < 0$, backward scattered. If $(\sigma+gn)^2$ is greater than k_o^2, for some n, then the corresponding μ_n is imaginary and hence that ray and all rays defined by larger values of n are exponentially attenuated with distance from the boundary and hence contribute nothing to the sound field at large distances from the boundary; these attenuated waves are called inhomogeneous or evanescent waves. Note that if λ_b is less than λ_o, (I.e., $g > k$) all rays except n=0 (and possibly n=1, depending on θ_o) are inhomogeneous, i.e., the rough boundary reflects just as a plane boundary.

The unknown coefficients A_n and B_n are determined from the two boundary conditions which are continuity of pressure across the boundary and continuity of normal velocity.

$$\rho_o\varphi = \rho_1\varphi_1 \tag{IV.6}$$

$$\partial\varphi/\partial n = \partial\varphi_1/\partial n \tag{IV.7}$$

where $\varphi=\varphi_i+\varphi_r$ and n indicates the normal direction to the boundary. The equation of the boundary is $z=\zeta(x)$ and consequently Eq. IV.7 can be written

$$\frac{\partial(\varphi - \varphi_1)}{\partial z} - \frac{\partial\zeta}{\partial x}\frac{\partial(\varphi - \varphi_1)}{\partial x} = 0 \tag{IV.8}$$

At this point in the analysis, we must make the basic approximation Rayleigh[9] did. It consists in saying that the slope of the boundary is very small, i.e., $\partial\zeta/\partial x \approx 0$.

For $\zeta(x)=a\cos(2\pi x/\lambda_b)$, this means that $2\pi a/\lambda_b \approx 0$ or the amplitude of the undulations must be very much smaller than their wavelength. With this approximation and the definitions of the potentials in Eqs. IV.1, IV.2, and IV.4, the boundary conditions become (the limits of summation which are always $-\infty$ to $+\infty$ are omitted).

$$\sum B_n \exp\{i[(\sigma_o+gn)x-\chi_n z]\}\Big|_{z=\zeta(x)} = \frac{\rho_o}{\rho_1}\Big[\exp\{i(\sigma_o x-\mu_o z)\} + \sum A_n \exp\{i[(\sigma_o+gn)x+\mu_n z]\}\Big]_{z=\zeta(x)} \qquad (IV.9)$$

$$\Big[-i\mu_o \exp\{i(\sigma_o x-\mu_o z)\} + i\sum A_n\mu_n \exp\{i[(\sigma_o+gn)x+\mu_n z]\} + i\sum B_n\chi_n \exp\{i[(\sigma_o+gn)x-\chi_n z]\}\Big]_{z=\zeta(x)} = 0 \qquad (IV.10)$$

At this point in the analysis we must introduce another approximation which naturally restricts the applicability of the results but which makes the solution of Eqs. IV.9 and IV.10 tractable. This approximation consists of restricting our attention to high frequencies such that $\lambda_o \ll \lambda_b$. It is then true that $gn \ll \sigma \leq k$. We also assume that the wavelength in the transmitted medium is much less than λ_b so that $gn \ll k_1$. It is also evident that for these inequalities to hold we must restrict our attention to rather small values of n, i.e., to the first one or two rays on either side of the principal (n=0) ray. However, as noted earlier, the rays defined by large n are quite likely to be inhomogeneous and thus contribute nothing to the sound field at a large distance from the boundary.

With the approximations just described, we can write

$$\mu_n \approx \sqrt{k_o^2-\sigma_o^2} = k_o\cos\theta_o \qquad (IV.11a)$$

$$\chi_n \approx \sqrt{k_1^2-\sigma_o^2} = k_1\cos\theta_1 \qquad (IV.11b)$$

Snell's law, $(\sin\theta_1)/c_1=(\sin\theta_o)/c_o$ was used to obtain Eq. IV.11b. We can now bring μ_n and χ_n out from under the summation signs in Eq. IV.10.

In this report, we are primarily concerned with determining what is reflected from the boundary and hence we will only solve for the coefficients A_n, although

there is no difficulty in solving for the transmitted rays as well.* We use Eq. IV.9 to eliminate the coefficients B_n from Eq. IV.10 which, after a little algebra, can be written

$$-\left(\frac{Z_1-Z_0}{Z_1+Z_0}\right) \exp[-i\ 2k_o\zeta(x)\ \cos\theta] + \sum A_n \exp[ignx] = 0 \tag{IV.12}$$

where

$$Z_1 = \frac{\rho_1 c_1}{\cos\theta_1} \quad \text{and} \quad Z_0 = \frac{\rho_o c_o}{\cos\theta_o} \tag{IV.13}$$

The first factor in the first term of Eq. IV.12 is of course the plane boundary reflection factor which we denote by R. Employing the standard identity

$$\exp[-i\ \alpha\ \cos(gx)] = \sum_{n=-\infty}^{\infty} (-i)^n\ J_n(\alpha)\ \exp[ignx] \tag{IV.14}$$

in Eq. IV.12, where J_n is the Bessel function of the first kind of order n, we finally obtain this result.

$$R \sum (-i)^n\ J_n(2k_o a\ \cos\theta_o)\ \exp[ignx] = \sum A_n\ \exp[ignx]. \tag{IV.15}$$

It is now a trivial task to obtain the coefficients A_n by equating terms with the same x-dependence [note that $J_{-n}=(-1)^n J_n$].

$$A_o = RJ_o(2k_o a\ \cos\theta_o) \qquad A_2 = -RJ_2(2k_o a\ \cos\theta_o)$$

$$A_1 = -iRJ_1(2k_o a\ \cos\theta_o) \qquad A_{-2} = -RJ_2(2k_o a\ \cos\theta_o)$$

$$A_{-1} = -iRJ_1(2k_o a\ \cos\theta_o) \qquad \text{etc.}$$

These results are very similar to those obtained by Rayleigh for the case of normal incidence; in fact Rayleigh,[9] before leaving the problem of scattering from a rough boundary, gives the above result for the amplitude of the specularly reflected ray (i.e., n=0). It is noted that the amplitude of the corresponding rays on either side of the specular direction are precisely equal, i.e., $A_1=A_{-1}$. The results Rayleigh quotes for normal incidence are presumably the total

*In Sec. B for illustrative purposes we actually solve for other than the reflected waves.

amplitude of the pair of corresponding rays since his A_1 is twice the result given here and he quotes no result for A_{-1}. For large argument, all Bessel functions look like damped sinusoids; therefore, with increasing frequency, less and less sound is reflected in the specular and near-specular directions. This is one of the basic features of the transient response. Furthermore the most damaging component of the pressure wave is associated with the specular reflection.

We turn now to the consideration of scattering of a plane wave incident from a fluid onto other media. It is stressed that throughout the remainder of the report, the two basic approximations already encountered, i.e., that the slope of the boundary is small and that the wavelengths of all reflected or refracted compressional or shear waves are small compared to the wavelength of the boundary, will be employed. Because the analyses are very similar to the preceding, their presentation will be quite brief.

B. Scattering from a Sinusoidal Boundary between a Semi-Infinite Fluid Medium and a Semi-Infinite Elastic Medium

The geometry of the problem is essentially that of Fig. 3 except that we now assume the upper medium to have a shear wave velocity b_1 and, correspondingly, a set of shear waves defined by the potential ψ_1

$$\psi_1 = \sum_{n=-\infty}^{\infty} C_n \exp\{i[(\sigma_o+gn)x-J_n z]\} \tag{IV.16}$$

$$\nu_n^2 = \kappa_1^2 - (\sigma_o+gn)^2 \tag{IV.17}$$

where

$$\kappa_1 = \omega/b_1 \quad .$$

The boundary conditions are continuity of normal displacement, continuity of normal stress and vanishing of the tangential stress. In terms of the potentials φ $(=\varphi_i+\varphi_r)$, φ_1 and ψ_1, these become[11] (assuming $\partial\zeta/\partial x \approx 0$)

$$\left.\begin{aligned}
&-\frac{\partial\varphi}{\partial z} + \frac{\partial\varphi_1}{\partial z} + \frac{\partial\psi_1}{\partial x} = 0 \\
&-\frac{\rho_o}{\rho_1}\varphi + \varphi_1 - \frac{2}{\kappa^2}\left(\frac{-\partial^2\varphi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial x\partial y}\right) = 0 \\
&\frac{\partial^2\varphi_1}{\partial x\partial y} - \frac{\partial^2\psi_1}{\partial z^2} + \frac{\partial^2\psi_1}{\partial xz} = 0
\end{aligned}\right\} \text{ at } z=\zeta(x) \tag{IV.18}$$

The assumption that all wavelengths are small compared to that of the boundary allows us to write, as before,

$$\mu_n \approx k_o \cos\theta_o$$

$$\chi_n \approx k_1 \cos\theta_1 \qquad \text{(IV.19a,b,c)}$$

$$\upsilon_n \approx k_1 \cos\gamma_1$$

The angles θ_1, and γ_1, are defined by Snell's law

$$\frac{\sin\theta_o}{c_o} = \frac{\sin\theta_1}{c_1} = \frac{\sin\gamma_1}{b_1} \qquad \text{(IV.20)}$$

where γ is the angle that the principal (n=0) ray of the set of transmitted shear waves makes with the z-direction.

For convenience, let us introduce the notation

$$A = \sum A_n \exp[i(gnx + \mu_n\zeta)]$$

$$B = \sum B_n \exp[i(gnx - \chi_n\zeta)] \qquad \text{(IV.21a,b,c)}$$

$$C = \sum C_n \exp[i(gnx - \upsilon_n\zeta)]$$

Whence the boundary conditions Eq. IV.18 become, after some algebraic manipulation

$$\begin{bmatrix} -\cos\theta_o & -\frac{c_o}{c_1}\cos\theta_1 & \frac{1}{2}\frac{c_o}{b_1}\frac{\sin 2\gamma_1}{\cos\gamma_1} \\ -\frac{\rho_o}{\rho_1} & \cos 2\gamma_1 & -\sin 2\gamma_1 \\ 0 & \frac{b_1}{c_o}\frac{\sin 2\gamma_1}{\cos\gamma_1}\cos\theta_1 & \cos 2\gamma_1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -\cos\theta_o \exp(-ik_o\zeta\cos\theta_o) \\ \frac{\rho_o}{\rho_1}\exp(-ik_o\zeta\cos\theta_o) \\ 0 \end{bmatrix} \qquad \text{(IV.22)}$$

Solving for A, we find

$$A = \exp(-ik_o\zeta\cos\theta_o)\left[\frac{Z_1\cos^2 2\gamma_1 + Z_t\sin^2 2\gamma_1 - Z_o}{Z_1\cos^2 2\gamma_1 + Z_t\sin^2 2\gamma_1 + Z_o}\right] \qquad \text{(IV.23)}$$

where

$$Z_t = \frac{\rho_1 b_1}{\cos\gamma_1} \qquad \text{(IV.24}$$

while Z_o and Z_1 were defined in Eq. IV.13.

The term in brackets is, again, simply the plane boundary reflection factor (denoted V in Brekhovskikh[17]). Therefore we write

$$A_n \exp(ignx) = V \exp[-i(2k_o a \cos\theta_o)\cos(gx)] \tag{IV.25}$$

which, with the identity IV.4, becomes

$$\sum_{n=-\infty}^{\infty} A_n \exp(ignx) = V \sum_{n=-\infty}^{\infty} (-i)^n J_n(2k_o a \cos\theta_o) \exp(ignx) \tag{IV.26}$$

whose solutions are

$$\begin{aligned} A &= V\, J_o(2k_o a \cos\theta_o) \\ A_{\pm 1} &= -i\, V\, J_1(2k_o a \cos\theta_o) \\ A_{\pm 2} &= -V\, J_2(2k_o a \cos\theta_o) \end{aligned} \tag{IV.27}$$

Note that if we solve Eq. IV.22 for B, we obtain

$$\sum B_n \exp(ignx) = W \exp\{-i[(k_o\cos\theta_o - k_1\cos\theta_1)\, a\, \cos gx]\} \tag{IV.28}$$

where W is the amplitude of the transmitted compressional wave in the case of a plane boundary.[18] Eq. IV.28 has the solutions

$$B_n = W\, (-i)^n J_n[(k_o\cos\theta_o - k_1\cos\theta_1)] \;. \tag{IV.29}$$

If we solve Eq. IV.22 for C, we obtain

$$\sum C_n \exp(ignx) = P \exp\{-i[(k_o\cos\theta_o - \varkappa_1\cos\gamma_1)a\, \cos gx]\} \tag{IV.30}$$

where P is the amplitude of the transmitted shear wave in the case of a plane boundary.[19] Eq. IV.30 has the solutions

$$C_n = P\, (-i)^n J_n[(k_o\cos\theta_o - \varkappa_1\cos\gamma_1)a] \tag{IV.31}$$

C. Scattering from the Sinusoidal Surface of a Thick Fluid Layer Sandwiched between Two Different Semi-Infinite Fluid Media

The geometry of the problem is illustrated in Fig. 4. In addition to the incident wave φ_1, the reflected wave in the incident medium φ_r and the transmitted wave in the second medium φ_1, we must allow for a reflected wave from the plane

surface of the intermediate layer (the surface at $z=-h$) which we denote φ_2 and for a wave transmitted into the third fluid media, which we denote φ_t. We write these two additional potentials in this fashion

$$\varphi_2 = \sum_{n=-\infty}^{\infty} D_n \mathrm{esp}\{i[\sigma_o+gn)x + \chi_n z]\} \tag{IV.32}$$

$$\text{where } \chi_n^2 = k_1^2-(\sigma_o+gn)^2 \tag{IV.33}$$

$$\varphi_t = \sum_{n=-\infty}^{\infty} E_n \exp\{i[(\sigma_o+gn)x - \eta_n z]\} \tag{IV.34}$$

$$\text{where } \eta_n^2 = k_2^2 - (\sigma_o+gn)^2 \tag{IV.35}$$

It is evident that even though we have a plane boundary, these two waves must be written in the same fashion as the infinite set of waves associated with a sinusoidal boundary to be able to match the boundary conditions; for example, if the plane boundary of the intermediate fluid were a free boundary the potentials φ_1 and φ_2 would obviously have to cancel each other and, given φ_1, (Eq. IV.4), this could only be possible if φ_2 has the form defined above.

The boundary conditions are again continuity of pressure and normal velocity at both $z=\zeta(x)$ and $z=-h$. Under the assumption of negligible boundary slope, these become

$$\left.\begin{aligned} \rho_o(\varphi_i + \varphi_r) &= \rho_1(\varphi_1 + \varphi_2) \\ \frac{\partial(\varphi_i + \varphi_r)}{\partial z} &= \frac{\partial(\varphi_1 + \varphi_2)}{\partial z} \end{aligned}\right\} \text{ at } z=\zeta(x) \tag{IV.36a,b}$$

$$\left.\begin{aligned} \rho_1(\varphi_1 + \varphi_2) &= \rho_2\varphi_t \\ \frac{\partial(\varphi_1 + \varphi_2)}{\partial z^2} &= \frac{\partial\varphi_t}{\partial z} \end{aligned}\right\} \text{ at } z=-h \tag{IV.37a,b}$$

As usual, the assumption that all wavelengths are small compared to that of the boundary allows us to approximate the direction cosines as in Eq. IV.19a,b and in addition to write

$$\eta_n \approx k_2\cos\theta_2 \tag{IV.38}$$

where θ_2 is defined by Snell's law

$$\frac{\sin\theta_o}{c_o} = \frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2} . \tag{IV.39}$$

Employing the shorthand notation introduced in Eq. IV.21 (where D and E are the right-hand sides of Eqs. IV.32 and IV.33, less the factor exp(ipz), the four boundary conditions can be cast in the form

$$\begin{bmatrix} -\rho_o e^{ik_o\zeta\cos\theta_o} & \rho_1 e^{-ik_1\zeta\cos\theta_1} & \rho_1 e^{ik_1\zeta\cos\theta_1} & 0 \\ k_o\cos\theta_o e^{ik_o\zeta\cos\theta_o} & k_1\cos\theta_1 e^{-ik_1\zeta\cos\theta_1} & -k_1\cos\theta_1 e^{ik_1\zeta\cos\theta_1} & 0 \\ 0 & \rho_1 e^{ik_1 h\cos\theta_1} & \rho_1 e^{-ik_1 h\cos\theta_1} & -\rho_2 e^{ik_2 h\cos\theta_2} \\ 0 & k_1\cos\theta_1 e^{-ik_1 h\cos\theta_1} & -k_1\cos\theta_1 e^{-ik_1 h\cos\theta_1} & -k_2\cos\theta_2 e^{ik_2 h\cos\theta_2} \end{bmatrix} \begin{bmatrix} A \\ B \\ D \\ E \end{bmatrix} = \begin{bmatrix} \rho_o e^{ik_o\zeta\cos\theta_o} \\ k_o\cos\theta_o e^{-ik_o\zeta\cos\theta_o} \\ 0 \\ 0 \end{bmatrix} \tag{IV.40}$$

Solving for A, the amplitudes of the set of reflected waves in the incident medium we obtain

$$\sum_{n=-\infty}^{\infty} A_n e^{ignx} = e^{-i2k_o\zeta\cos\theta} \left\{ \frac{(-1+\frac{Z_2}{Z_0})\cos[k_1(h+\zeta)\cos\theta_1] - i(\frac{Z_1}{Z_0} - \frac{Z_2}{Z_1})\sin[k_1(h+\zeta)\cos\theta_1]}{(1+\frac{Z_2}{Z_0})\cos[k_1(h+\zeta)\cos\theta_1] - i(\frac{Z_1}{Z_0} + \frac{Z_2}{Z_1})\sin[k_1(h+\zeta)\cos\theta_1]} \right\} \tag{IV.41}$$

where

$$Z_2 = \frac{\rho_2 c_2}{\cos\theta_2} \tag{IV.42}$$

while Z_0 and Z_1 were defined in Eq. IV.13.

Now if ζ is negligible compared to h (i.e., if the amplitude of the boundary undulations are small compared to the thickness of the intermediate layer) so that we may discard it in the arguments of the sines and cosines in the factor in brackets

in Eq. IV.41, then that factor reduces to the standard plane-boundary reflection factor for the three-fluid layer problem.[20] We denote the factor by Ω.

Employing the Bessel function identity Eq. IV.14, Eq. IV.41 becomes

$$\sum_{n=-\infty}^{\infty} A_n e^{ignx} = \Omega \sum_{n=-\infty}^{\infty} (-i)^n e^{ignx} J_n(2k_o a \cos\theta_o) \tag{IV.43}$$

which has solutions

$$A_n = \Omega (-i)^n J_n(2k_o a \cos\theta_o) \tag{IV.44}$$

If one were to solve for the amplitudes of the various other sets of waves, B, D, or E, the results would quite evidently be the appropriate reflection or transmission factor from the plane boundary case multiplied by a Bessel function whose argument involves the difference of wave numbers as in Eq. IV.29 and IV.31.

D. Scattering from the Sinusoidal Surface of a Thick Elastic Layer Sandwiched between Two Different Semi-Infinite Fluid Media

The geometry of the problem is the same as in Fig. 4 except we now assume the intermediate layer to have the shear wave velocity b_1 and correspondingly, two sets of shear waves - one set that originates at the sinusoidal boundary and the other that represents reflections from the plane boundary.

It is evident from the preceding work that under the three basic assumptions made in this analysis, viz., negligible boundary slope, all wavelengths much less than the boundary wavelength and layer thickness much greater than amplitude of boundary undulations, the reflection or transmission factors for a sinusoidal boundary are simply the corresponding factor for a plane boundary multiplied by a Bessel function whose argument involves the wave numbers in the appropriate media, the amplitude of the boundary, and the angle of incidence. We shall therefore spare the reader's time and simply quote the answer for the reflection coefficients A_n of the reflected potential (Eq. IV.2),

$$A_n = \Gamma(-i)^n J_o(2k_o a \cos\theta_o)$$

where Γ is given in Ref. 21.

E. Scattering from a Sinusoidal Boundary between Two Semi-Infinite Fluid Media - Slope of Boundary Not Negligible

Let us return to the problem considered in Sec. A and relax the condition on the slope of the boundary, i.e., let us try to solve the problem using the exact form of the second boundary condition (Eq. IV.8). For ζ=a cosgx, this second boundary condition can be written

$$-\mu_o e^{-i\mu_o\zeta}+\sum A_n\mu_n e^{i(gnx+\mu_n\zeta)} + \sum B_n\chi_n e^{i(gnx-\chi_n\zeta)} =$$
$$\frac{-iag}{2}[e^{igx}-e^{-igx}]\{\sum B_n(\sigma_o+gn)e^{i(gnx-\chi_n\zeta)}-\sigma e^{-i\mu_o\zeta}-\sum A_n(\sigma_o+gn)e^{i(gnx+\mu_n\zeta)}\} \tag{IV.45}$$

Employing the approximation that all wavelengths are much smaller than that of the boundary so that $\mu_n\approx k_o\cos\theta_o$, $\chi_n\approx k_1\cos\theta_1$ and $\sigma_o+gn\approx k_o\sin\theta_o$ (note that because of this last approximation, theresults below will not be strictly valid for normal incidence), eliminating the B_n from Eq. IV.45 by the use of Eq. IV.9, and finally employing the Bessel function identity Eq. IV.14, we obtain this relationship for the amplitudes of the reflected rays

$$(Z_0-Z_1)\sum(-i)^n J_n(2k_oa\cos\theta_o)e^{ignx} + (Z_0+Z_1)\sum A_n e^{ignx} =$$
$$\frac{-iag}{2}[Z_1\tan\theta_o(\frac{\rho_o}{\rho_1}-1)](e^{igx}-e^{-igx})\sum[(-i)^n J_n(2k_oa\cos\theta_o)+A_n]e^{ignx} \tag{IV.46}$$

If we now equate terms with the same x-dependence on both sides of the equation, we generate an infinite set of coupled equations each of which involves three of the unknown A_n. For example, gathering the terms that have no x-dependence and that have x-dependence exp($\pm$igx) results in these three equations

$$A_0 = RJ(\alpha) + \frac{\beta}{Z_1+Z_0}[-A_1+A_{-1}]$$
$$A_1 = -iRJ_1(\alpha) + \frac{\beta}{Z_1+Z_0}[J_0(\alpha)+J_2(\alpha)+A_0-A_2] \tag{IV.47a,b,c}$$
$$A_{-1} = -iRJ_1(\alpha) + \frac{\beta}{Z_1+Z_0}[-J(\alpha)-J_2(\alpha)-A_0+A_{-2}]$$

where

$\alpha = 2k_0a\cos\theta_o$ and $\beta = (-iag/2)[Z_1\tan\theta_o(\rho_0/\rho_1-1)]$.

In general, the nth coefficient A_n is coupled to both A_{n-1} and A_{n+1}. We solve these by iteration, i.e., we use the results obtained in Sec. A (now denoted $A_n^{(0)}$ in the right-hand side of these equations, to generate a first order correction (denoted $A_n^{(1)}$ which includes the slope of the boundary. Noting that $A_{-n}^{(0)}=A_{+n}^{(0)}$, we obtain

$$A_0^{(1)} = RJ(\alpha)$$

$$A_{+1}^{(1)} = -iRJ_1(\alpha) + \beta \frac{(2Z_1)}{(Z_1+Z_0)^2} [J_0(\alpha) + J_2(\alpha)] \qquad \text{(IV.48a,b,c)}$$

$$A_{-1}^{(1)} = -iRJ_1(\alpha) - \beta \frac{(2Z_1)}{(Z_1+Z_0)^2} [J_0(\alpha) + J_2(\alpha)]$$

Employing the identity

$$J_{n-1}(\alpha) + J_{n+1}(\alpha) = 2nJ_n(\alpha)/\alpha \quad , \qquad \text{(IV.49)}$$

the last two equations can be written

$$A_{\pm1}^{(1)} = -iRJ_1(2k_o a \cos\theta_o)\{1\mp \frac{g}{k_o\cos\theta_o} \frac{[1-(\rho_0/\rho_1)]}{[1-(Z_0/Z_1)^2]} \tan\theta_o \qquad \text{(IV.50)}$$

In general

$$A_{\pm n}^{(1)} = (-i)^n RJ_n(2k_o a \cos\theta_o)\{1\mp \frac{ng}{k_o\cos\theta_o} \frac{[1-(\rho_0/\rho_1)]}{[1-(Z_0/Z_1)^2]} \tan\theta_o \qquad \text{(IV.51)}$$

The correction term is seen to be independent of a, the amplitude of the boundary undulations. Also the correction term has destroyed the symmetry of the amplitude coefficients so that A_{-n} no longer equals A_n.

V. THE REFLECTION OF PLANE WAVES FROM A LIQUID-SATURATED POROUS ELASTIC HALF-SPACE

In an earlier report[1] formulas were derived for the reflection coefficient for a plane harmonic wave incident on a porous fluid saturated elastic solid using field equations derived by Biot.[22] The reflection coefficient so formulated depends on seven parameters characterizing the liquid-saturated porous elastic solid as well as the density and sound speed of the fluid in which the plane wave

propagates. The seven parameters characterizing the porous elastic solid are:

1. ρ_s, the density of the pure compacted solid.
2. ρ_f, the density of the fluid which saturates the solid.
3. C_u, the unjacketed compressibility of the liquid saturated elastic solid.
4. C_j, the jacketed compressibility.
5. N, the shear modulus of the material.
6. χ, the structure factor.
7. β, the porosity.

The physical significance of the above parameters are described in the report by Eichler and Rattayya.[1]

Certain of the above parameters can be determined from experiments while the others must be estimated. For example, the only measurements made of unjacketed and jacketed compressibility were those made by Fatt.[23] However these were made for a kerosene saturated sandstone under static conditions in the laboratory and in any case would not be applicable to a theory which attempts to model the ocean bottom. A similar situation exists with regard to χ, the structure factor; a factor χ approximately equal to 4.3 has been measured for an air-filled sand.[24] Very few measurements of the shear wave velocity in ocean bottom sediments are available. In fact the only ones we are aware of are those reported by Laughton.[25]

With the above limitations in mind, a numerical procedure to determine the reflection coefficient as a function of the grazing angle has been developed and programmed for a digital computer. Assumed values for the structure factor χ and the unjacketed compressibility C_u together with experimentally obtained values of the porosity β, saturating fluid density ρ_f (in all cases the saturating fluid density is that of sea water) the bulk density ρ (mass per unit volume of the combined solid and liquid), the dilatational wave speed V_p, and the shear wave speed V_s, are entered into the program. The program then proceeds to calculate the quantities necessary for the determination of the reflection coefficient.

In each case the calculations were performed for a relatively large structure factor χ=11 and two smaller structure factors χ=1, and χ=2. The data used was that of Laughton on Globigerina Ooze.

In both Figs. 5 and 6 we show plots of the bottom loss per reflection versus grazing angle for a given set of parameters. The bottom loss per reflection is defined by $[-20 \log|R|]$ where $|R|$ is the modulus of the reflection coefficient.

In each figure we show two plots, one for a value of C_u=.1 C_f and the other for C_u=.2 C_f, where C_f is the compressibility of the saturating fluid. Fig. 7 shows the bottom loss per reflection for a given set of parameters for two different values of the structure factor χ=1, and χ=2. It is characteristic of each of these plots that there is some angle at which the bottom loss is a maximum. There is also some angle above which the bottom loss is relatively insensitive to change in the grazing angle. Fig. 8 shows a plot of the phase angle φ of the reflection coefficient superimposed on a plot of bottom loss. From this figure we see that the angle above which the bottom loss is fairly constant is precisely the critical angle θ_c, where θ_c is given by $\theta_c=\cos^{-1}(c_o/V_p)$, c_o is the sound speed in the fluid and V_p is the dilatational velocity in the solid (i.e., the angle for which the phase angle of the reflection coefficient goes to zero).

Since we have neglected viscosity in the above calculations the reflection coefficients obtained are independent of frequency.

APPENDIX A: Laplace Transform of Time Dependent Green's Function for a Wedge with Pressure-Release Walls

Consider the problem of a source with Dirac delta time dependence located at the point $S(r_o,\theta_o)$ within a wedge-shaped region $0 \le \theta \le \alpha$, the boundaries of which are pressure-release surfaces. The pressure must therefore satisfy the equation:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\delta(r-r_o)\delta(\theta-\theta_o)\delta(t)}{r} \tag{A.1}$$

and the boundary conditions

$$\begin{aligned} p(r,0) &= 0 \\ p(r,\alpha) &= 0 . \end{aligned} \tag{A.2}$$

Taking a Laplace Transform in time yields the following system of equations:

$$\frac{\partial^2 \bar{p}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{p}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{p}}{\partial \theta^2} - \frac{s^2}{c^2}\bar{p} = -\frac{\delta(r-r_o)\delta(\theta-\theta_o)}{r} \tag{A.3}$$

and

$$\begin{aligned} \bar{p}(r,0) &= 0 \\ \bar{p}(r,\alpha) &= 0 \end{aligned} \tag{A.4}$$

where $\bar{p}(r,\theta)$ is the Laplace Transform of $p(r,\theta,t)$.

The angular eigenfunctions corresponding to the boundary conditions (B.4) are

$$\Phi_n(\theta) = A_n \sin n\theta \tag{A.5}$$

where

$$\nu_n = \frac{n\pi}{\alpha}\ , \ n=1,2,3\ \dots \tag{A.6}$$

Assuming that the transformed pressure $\bar{p}$ can be expanded in the form

$$\bar{p}(r,\theta;s) = \sum_{n=1}^{\infty} A_n(r;s) \sin \nu_n\theta \tag{A.7}$$

and using the orthogonality of the eigenfunctions over the range $0 < \theta < \alpha$ it is found that the functions A_n must satisfy

$$\frac{d^2A_n}{dr^2} + \frac{1}{r}\frac{dA_n}{dr} - \left(\frac{s^2}{c^2} + \frac{\nu_n^2}{r^2}\right) A_n = -\frac{2}{\alpha} \sin \nu_n\theta_o \frac{\delta(r-r_o)}{r} \tag{A.8}$$

Equation A.8 is solved using Hankel Transforms and the result is

$$A_n(r,s) = \frac{2}{\alpha} \sin \nu_n\theta_o\, I_{\nu_n}\left(\frac{s}{c}\, r <\right) K_{\nu_n}\left(\frac{s}{c} r >\right) \tag{A.9}$$

where $r <$ and $r >$ are, respectively, the smaller or larger of the quantities r and r_o.

Using the identidy

$$\sin \nu_n\theta_o \sin \nu_n\theta = \frac{1}{2}\{\cos\nu_n(\theta-\theta_o) - \cos\nu_n(\theta+\theta_o)\} \tag{A.10}$$

the solution to the set of equations A.3 and A.4 can be written as

$$\bar{p}(r,\theta;s) = \frac{1}{2\alpha}\sum_{n=0}^{\infty} \epsilon_n[\cos\nu_n(\theta-\theta_o)-\cos\nu_n(\theta+\theta_o)]\, I_{\nu_n}\left(\frac{s}{c}\, r <\right)K_{\nu_n}\left(\frac{s}{c}\, r >\right) \tag{A.11}$$

where $\epsilon_o=1$ and $\epsilon_n=2$ for $n \geq 1$.

APPENDIX B: Asymptotic Evaluation of I

Consider the integral I where I is of the form

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\xi)\, e^{i(\xi x + z\sqrt{k_o^2-\xi^2})}\, d\xi \tag{B.1}$$

where the integral is taken along the real axis in the ξ-plane and the branch of $\sqrt{k_o^2-\xi^2}$ is that for which $\mathrm{Im}\sqrt{k_o^2-\xi^2} > 0$ (see Fig. 9a).

We define a conformal transformation $\xi = k_o \sin\varphi$ from the ξ-plane to the region $-\frac{\pi}{2} < \mathrm{Re}\varphi < \frac{\pi}{2}$ of the φ-plane. Then

$$\left.\begin{aligned} I &= \frac{1}{2\pi} \int_C A(k_o \sin\varphi)\, e^{ik_o(x\sin\varphi + z\cos\varphi)}\, k_o \cos\varphi\, d\varphi \\ &= \frac{k_o}{2\pi} \int_C A(k_o \sin\varphi)\, e^{ik_o \cos(\varphi-\theta)}\, \cos\varphi\, d\varphi \end{aligned}\right\} \tag{B.2}$$

where $x = r\sin\theta$, and $z = r\cos\theta$, and where the contour C is as shown in Fig. 9b. This integral may be evaluated asymptotically for the case where $k_o r \gg 1$ by the method of steepest descents. Letting

$$F(\varphi) = \cos\varphi (k_o \sin\varphi) \tag{B.3}$$

we have

$$I = \frac{k_o}{2\pi} \int_C F(\varphi)\, e^{ik_o r\cos(\varphi-\theta)}\, d\varphi \quad . \tag{B.4}$$

The location of the saddle point is found from the condition

$$\left[\frac{d}{d\varphi}(ik_o r\cos(\varphi-\theta))\right] = 0 \tag{B.5}$$

so that the saddle point (SP) is at $\varphi = 0$. The path of steepest descent (PSD) through the saddle point is given by

$$\left.\begin{aligned} & \mathrm{Im}\,\{ik_o r\, \cos(\varphi-\theta) - ik_o r\} = 0 \\ \text{i.e.,}\quad & \mathrm{Re}[\cos(\varphi-\theta)-1] = 0 \\ \text{or}\quad & \cos(u-\theta)\, \cosh v = 1 \end{aligned}\right\} \tag{B.6}$$

if $\varphi = u + iv$.

This path is sketched in Fig.9b. We deform the contour C into the PSD and, since the integrand decays rapidly away from the saddle point on this contour, we approximate the integral by its value along a small section of the path around the saddle point. The leading term of the asymptotic expansion for I is then given as

$$I \sim \sqrt{\frac{k_o}{2\pi r}} \cos\theta \; A(k_o \sin\theta) \; e^{i(k_o r-\pi/4)} \tag{B.7}$$

provided $\theta \neq \pi/2$, and $A(k_o \sin\theta)$ is bounded and non-zero. To this must be added the residues of the integrand at all poles which lie between the original path C and the steepest descent path (a possible set of these poles are shown circled in the figure). The poles are given by the zeros of the transcendental equation

$$i \tan \gamma_1 h = \frac{\rho_o \gamma_1}{\rho_1 \gamma_o} \qquad \text{where } \gamma_1 = \sqrt{k_1^2-\xi^2} \, . \tag{B.8}$$

Since $A_1(\xi)$ is an even function of γ_1, there are no branch cuts necessary to make γ_1 single-valued, so long as h is finite. As $h \to \infty$ the poles become more and more numerous along a line which in the limit becomes a branch line.

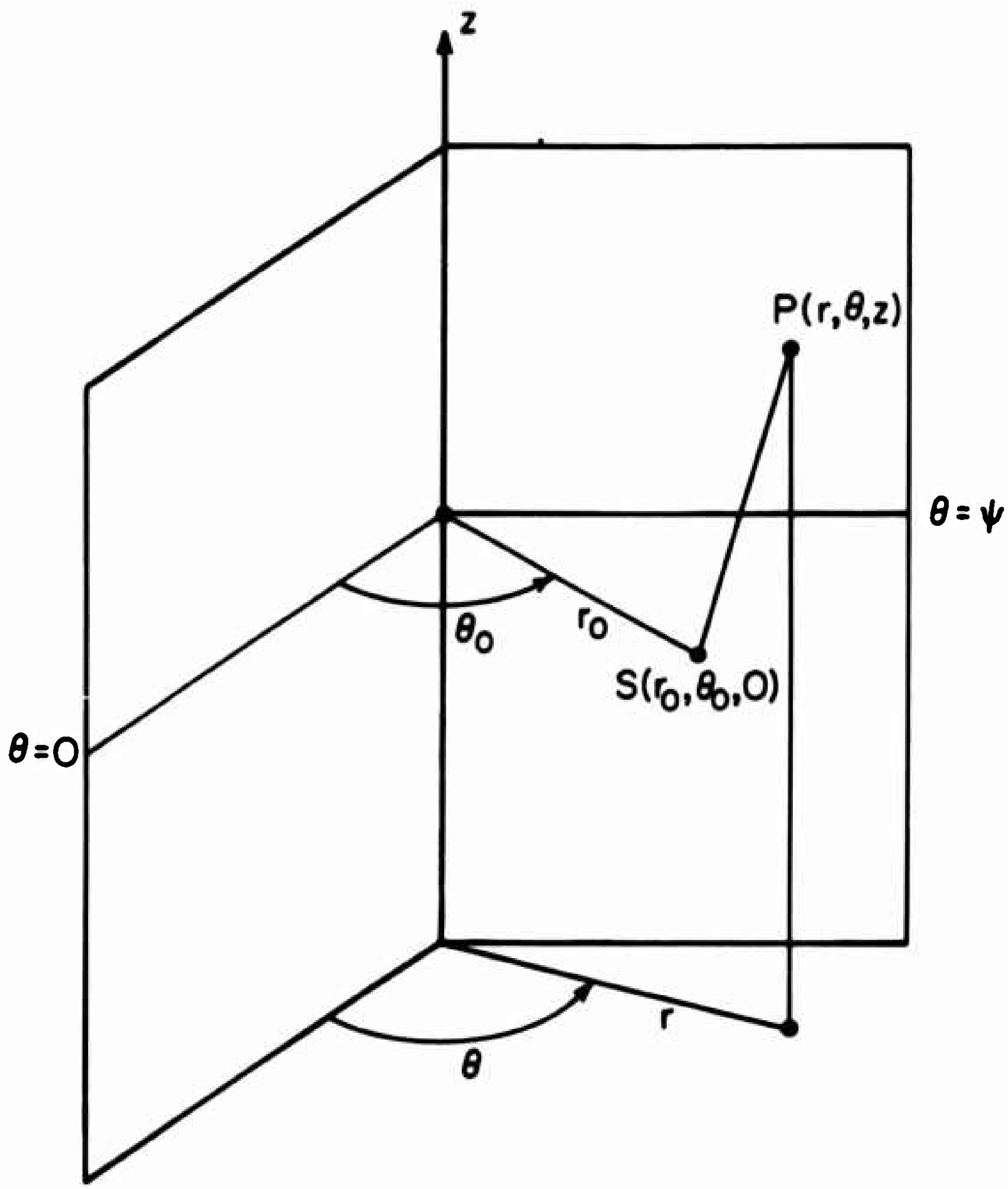


Fig. 1. Geometry of the problem associated with a point source at $S(r_o, \theta_o, 0)$ located within a wedge whose boundaries are given by $\theta=0$, $\theta=\psi$.

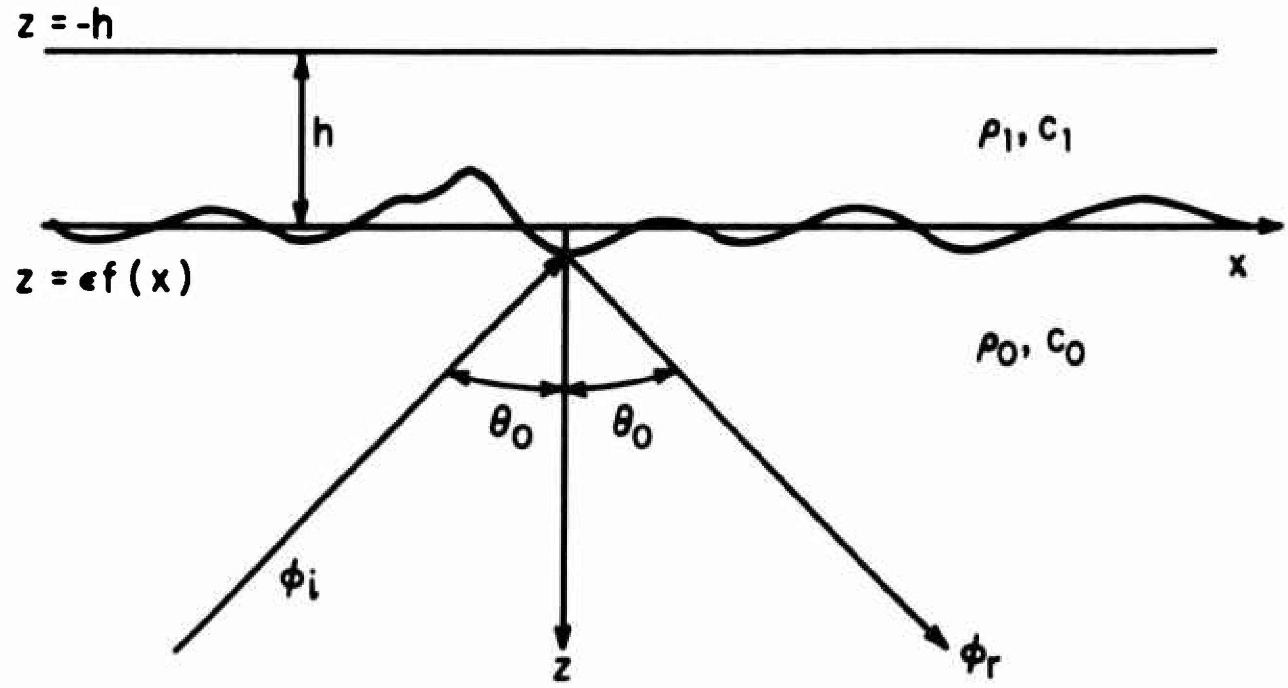


Fig. 2. Geometry of the problem associated with a plane wave φ_1 incident on an arbitrary rough interface $z=\epsilon f(x)$.

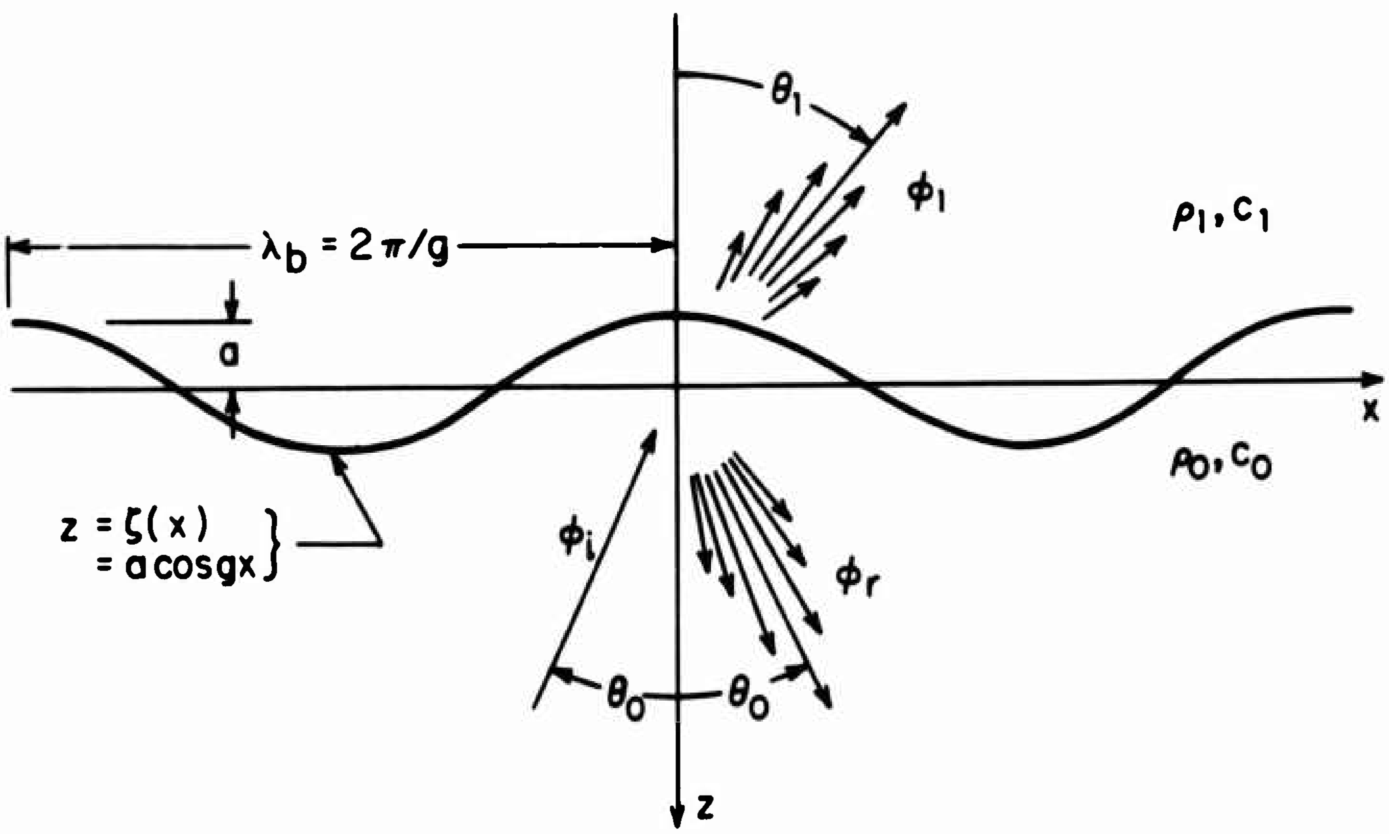


Fig. 3. Geometry of the problem associated with a plane sound wave obliquely incident onto the sinusoidally rough boundary between two semi-infinite fluid media.

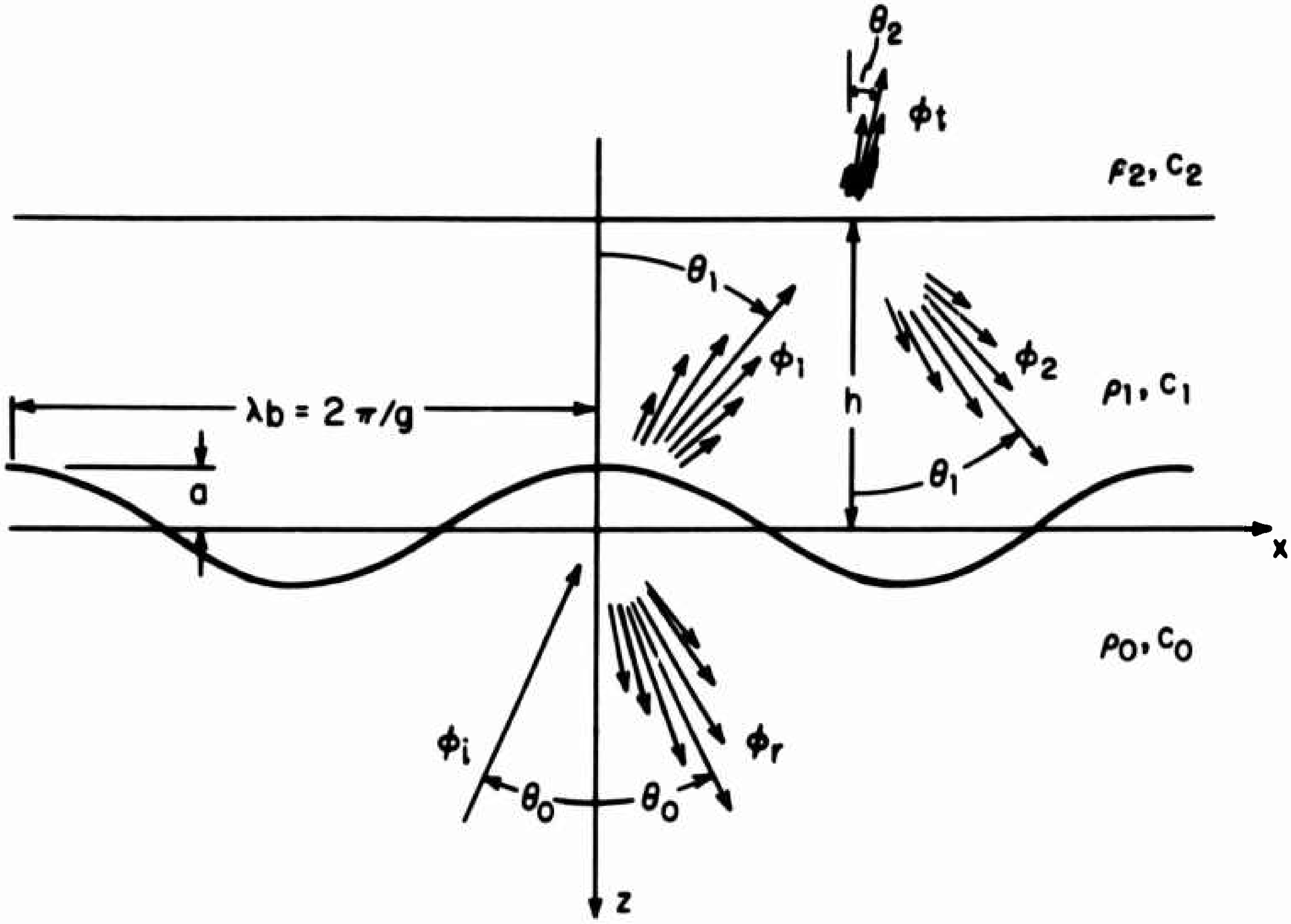


Fig. 4. Geometry of the problem associated with a plane sound wave obliquely incident onto the sinusoidal boundary of a thick fluid layer sandwiched between two different semi-infinite fluids.

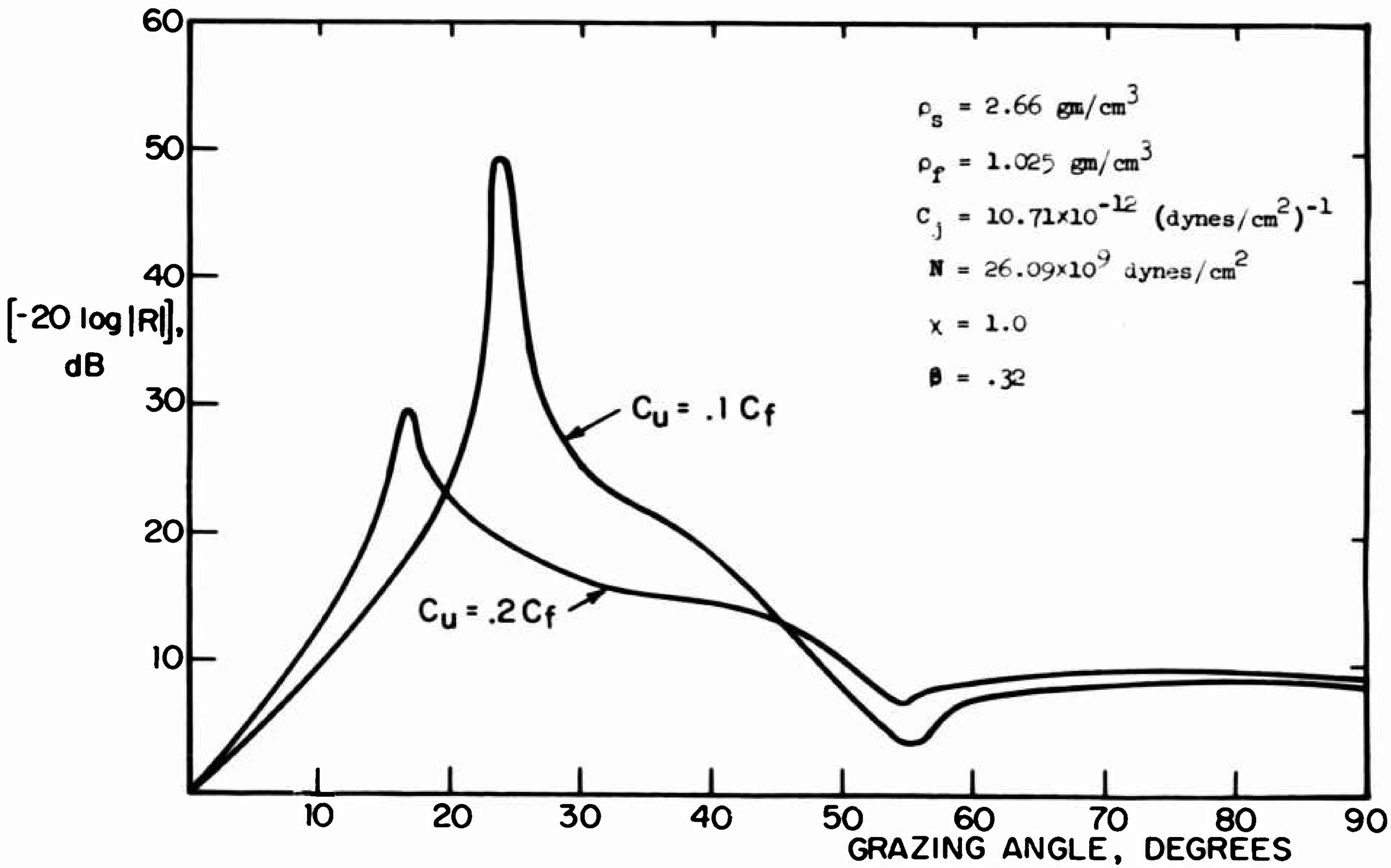


Fig. 5. Bottom loss per reflection in dB as a function of grazing angle for two values of the unjacketed compressibility C_u

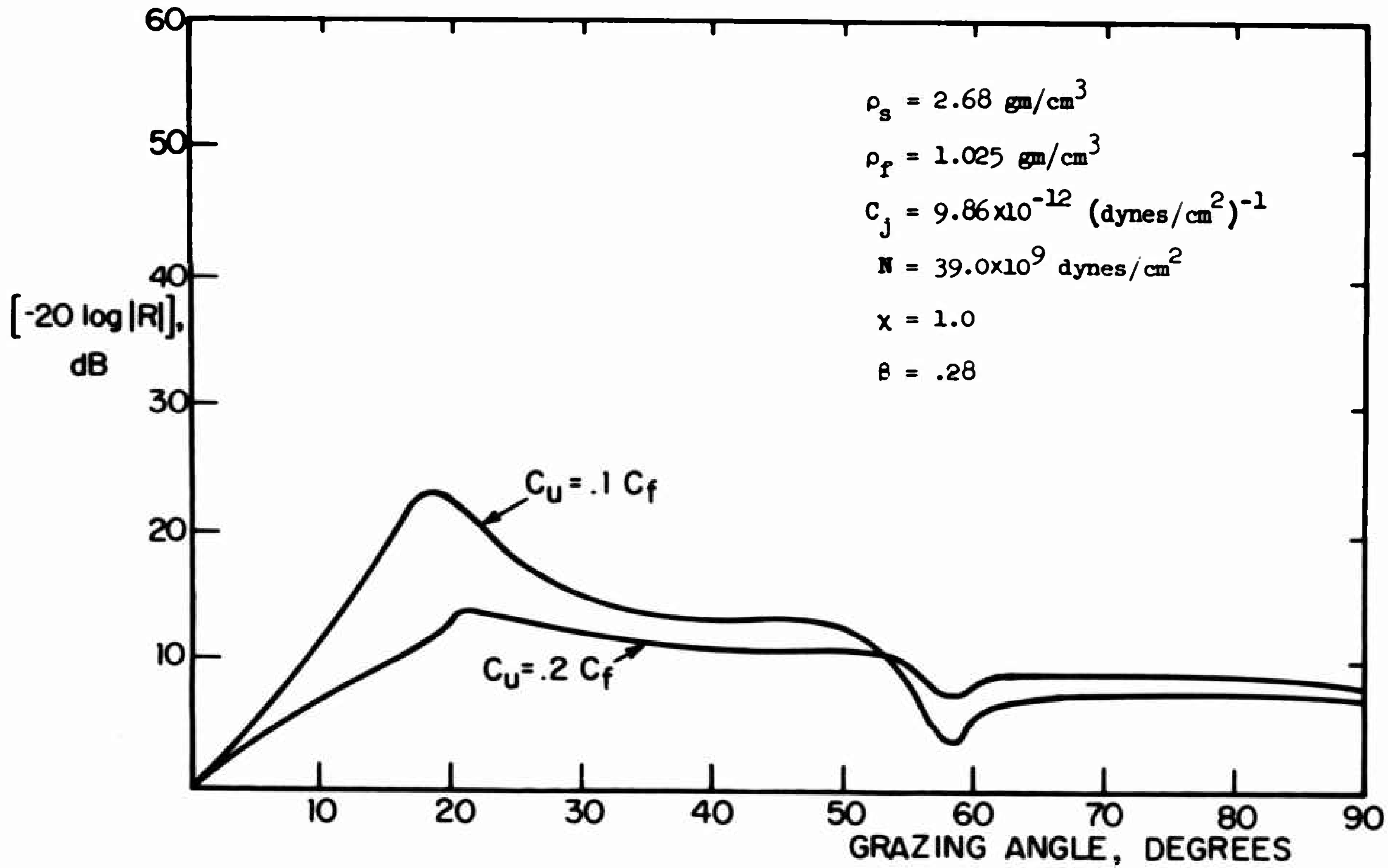


Fig. 6. Bottom loss per reflection in dB as a function of grazing angle for two values of the unjacketed compressibility C_u.

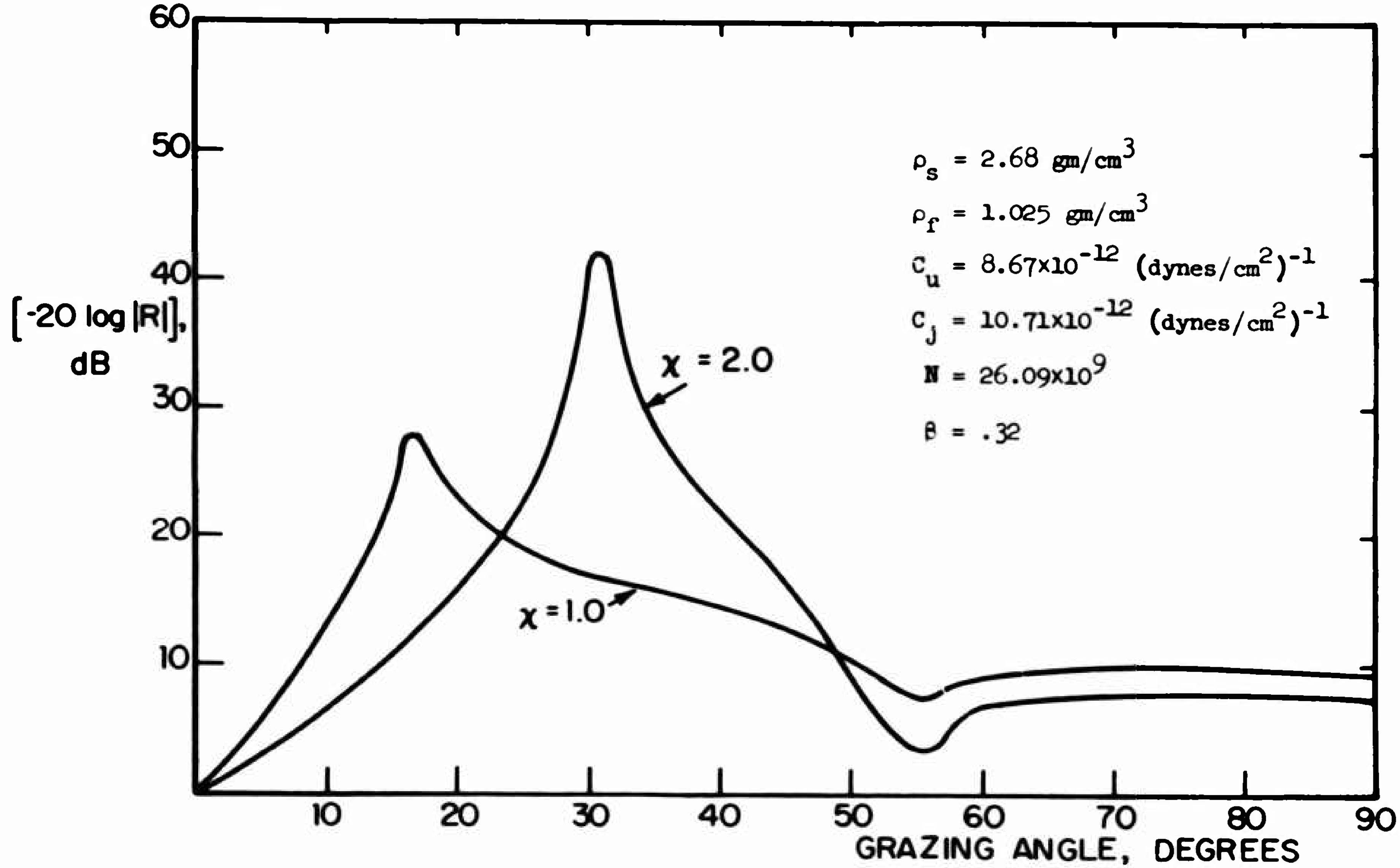


Fig. 7. Bottom loss per reflection in dB as a function of grazing angle for two values of the structure factor χ.

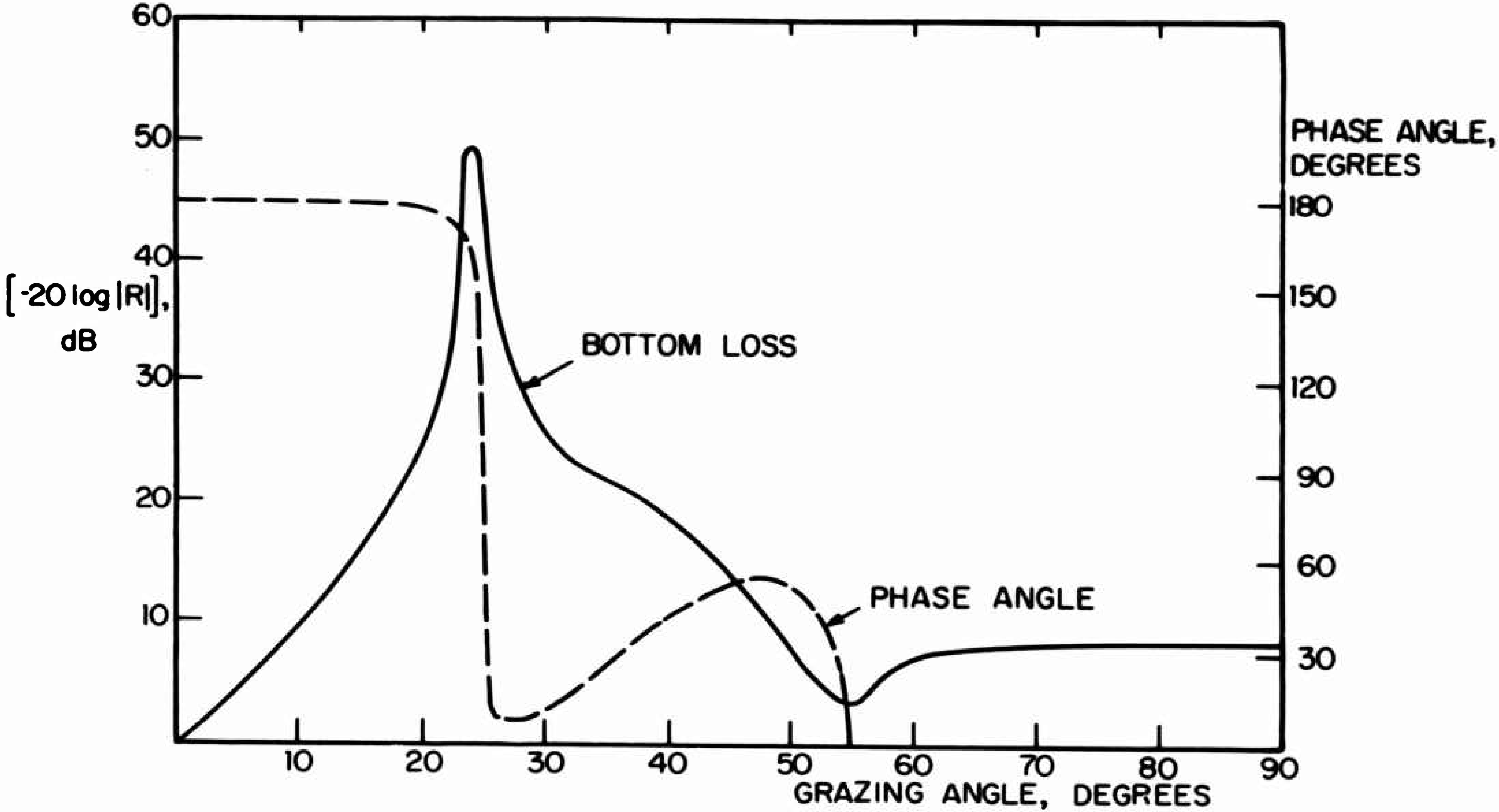
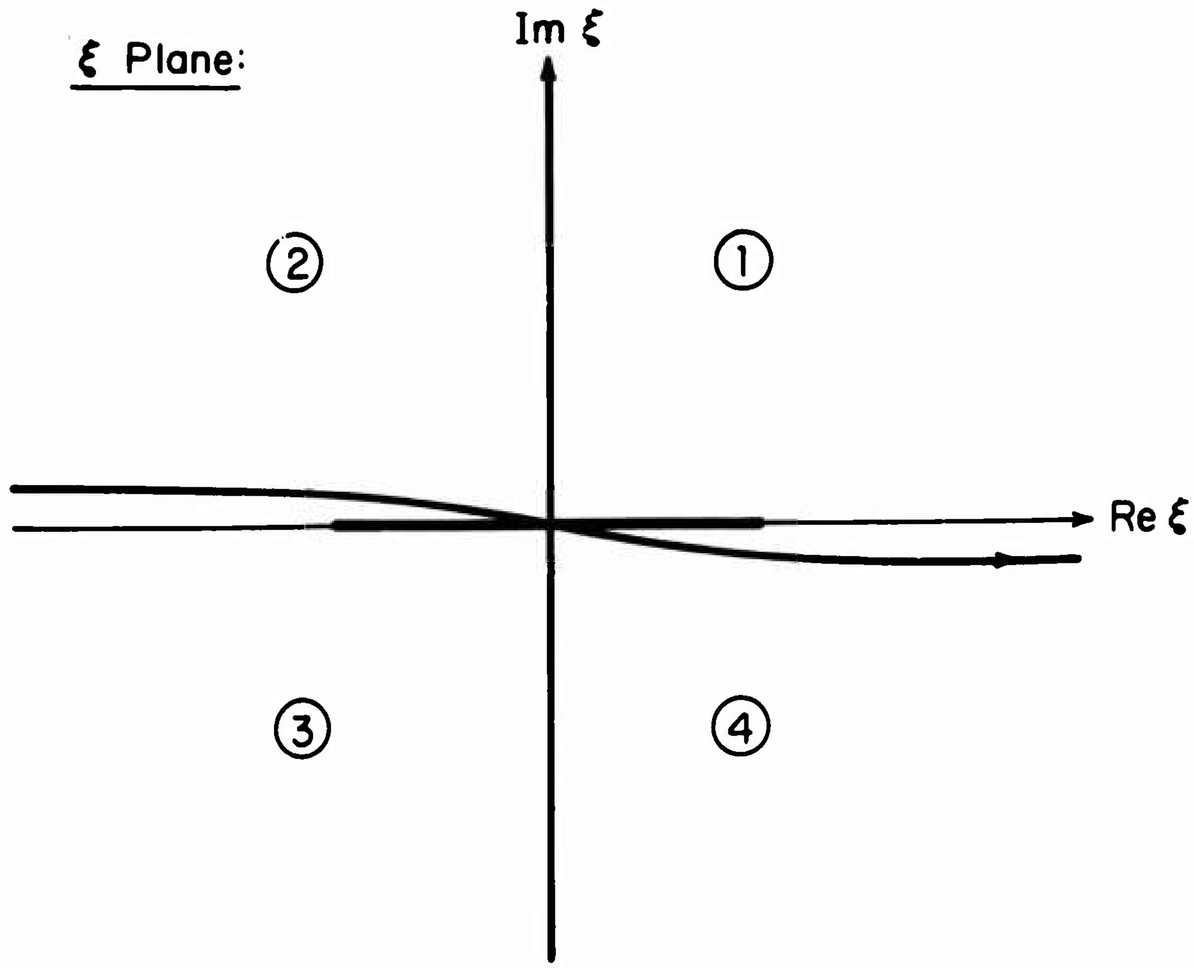
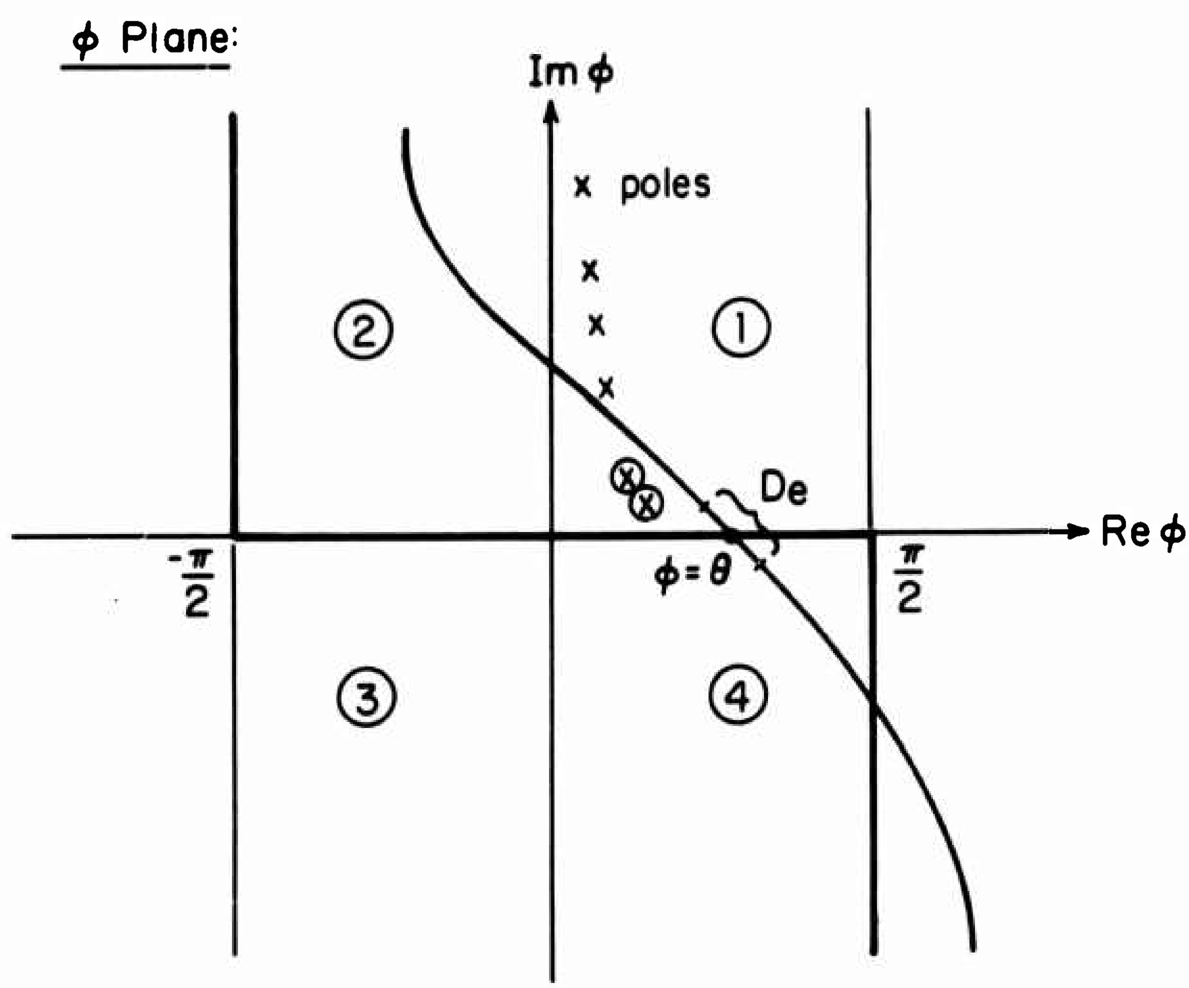


Fig. 8. Bottom loss in dB and phase angle of reflection coefficient as a function of grazing angle.



(a)



(b)

Fig. 9. (a) Integration path for evaluation of integral in Eq. B.1 in the complex ξ-plane. (b) Integration path in the complex φ-plane for evaluation of integral in Eq. B.2, also showing steepest descent path as well as possible pole locations.

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13. ABSTRACT

Various problems concerning the effects of the boundaries of the ocean on the propagation of pressure waves in the ocean are considered. The propagation of a transient pressure wave in a wedge shaped region of fluid is treated. This is the model chosen to describe the situation in which an underwater explosion takes place in a coastal ocean region which is characterized by a strongly sloping bottom. In an attempt to study the effects of the polar ice cap on the propagation of a pressure wave, the reflection of a plane wave onto a rough boundary separating a fluid half space and a thick fluid layer of differing sound speed and density is considered. These results are currently being used to construct the response to a transient pressure pulse and to generate numerical results for conditions representative of underwater explosions. The final chapter presents numerical values of the reflection coefficient as a function of grazing angle for the case of a plane wave incident on a porous elastic bottom. The analytical expressions used were derived in an earlier report of this series.

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13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

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